Homework 2

1. Of what category is the set of all rational numbers $\mathbb{Q}$ in $\mathbb{R}$?

2. Show that the complement M^c of a meager subset M of a complete metric space X is non-meager.

3. Let X and Y be Banach spaces. Suppose that $T_n \in B(X,Y)$ is such that $\sup_n \|T_n\| = +\infty$. Show that there is a point $x_0 \in X$ such that $\sup_n \|T_n x_0\|_Y = +\infty$. The point x_0 is called a point of resonance.

4. If X and Y are Banach spaces and $T_n \in B(X,Y)$, show that the following are equivalent:

(a) $\{\|T_n\|\}$ is bounded;

(b) $\{\|T_n x\|\}$ is bounded for all $x \in X$;

(c) $\{|g(T_n x)|\}$ is bounded for all $x \in X$ and for all $g \in Y^*$.

5. If the inverse T^{-1} of a closed linear operator exists, show that T^{-1} is a closed linear operator.

6. Let T be a closed linear operator. If two sequences $\{x_n\}$ and $\{\tilde{x}_n\}$ in the domain of T both converge to the same limit x, and if $\{Tx_n\}$ and $\{T\tilde{x}_n\}$ both converge, show that $\{Tx_n\}$ and $\{T\tilde{x}_n\}$ both have the same limit.

7. Let X and Y be Banach spaces and $T : X \to Y$ an injective bounded linear operator. Show that $T^{-1} : \operatorname{Ran} T \to X$ is bounded if and only if $\operatorname{Ran} T$ is closed in Y.

8. Let $T : X \to Y$ be a bounded linear operator, where X and Y are Banach spaces. If T is bijective, show that there are positive real numbers a and b such that

$$a\|x\|_X \leq \|Tx\|_Y \leq b\|x\|_X \quad \forall x \in X.$$

9. Let X be the normed space whose points are sequences of complex numbers $x = \{\xi_j\}$ with only finitely many non-zero terms and norm defined by $\|x\| = \sup_j |\xi_j|$. Let $T : X \to X$ be defined by

$$Tx = \left(\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \ldots\right) = \left(\frac{1}{j}\xi_j\right).$$

Show that T is linear and bounded, but T^{-1} is unbounded. Does this contradict the Bounded Inverse Theorem?

10. Let $\{T_n\}$ be a sequence of bounded linear transformations from $X \to Y$, where X and Y are Banach spaces. Suppose that $\|T_n\| \leq M < \infty$ and there is a dense set $E \subset X$ such that $\{T_n x\}$ converges for all $x \in E$. Show that $\{T_n x\}$ converges on all of X.

11. Let X and Y be normed spaces, $T \in B(X,Y)$ and $\{x_n\}$ a sequence in X. If $x_n \to x$ weakly, show that $Tx_n \to Tx$ weakly.

12. If $\{x_n\}$ is a weakly convergent sequence in a real normed space X with $x_n \to x_0$. Show that there is a sequence $\{y_m\}$ of linear combinations of elements of $\{x_n\}$ which converge strongly to x_0. Hint: Prove that the weak closure of a convex set is the same as the strong closure; then apply it to $E = \text{span}\{x_n\}$.

13. Let $T_n \in B(X, Y)$. Show that $T_n \to T$ uniformly if and only if for every $\epsilon > 0$ there is an N, depending only on ϵ such that for all $n > N$ and all $x \in X$ of norm 1 we have

$$\|T_n x - Tx\|_Y < \epsilon.$$

14. If a normed space X is reflexive, show that X^* is reflexive.

15. Let X be a normed space and $\{f_n\}$ a sequence of linear functionals on X. Show

(a) If $f_n \to f$ weakly, then $f_n \to f$ weak* too;

(b) Show that if X is reflexive, then the converse holds.

16. A weak Cauchy sequence in a normed space is a sequence $\{x_n\}$ in X such that for every $f \in X^*$ the sequence $\{f(x_n)\}$ is Cauchy. Show that a weak Cauchy sequence is bounded.

17. A normed space X is said to be weakly complete if each weak Cauchy sequence in X converges weakly in X. If X is reflexive, show that X is weakly complete.

18. Put $f_n(t) = e^{int}$, $t \in [-\pi, \pi]$. Let $L^p = L^p(-\pi, \pi)$ with respect to Lebesgue measure. If $1 \leq p < \infty$, show that $f_n \to 0$ weakly in L^p, but not strongly.

19. A subset of a Banach space X is weakly bounded if for all $f \in X^*$, $\sup_{x \in S} |f(x)| < \infty$. Prove that S is weakly bounded if and only if it is strongly bounded, i.e., $\sup_{x \in S} \|x\|_X < \infty$.

20. If $\{x_n\} \subset \ell^1$ show that $\sum_{j=1}^{\infty} y(j)x_n(j) \to 0$ for every $y \in c_0$ if and only if $\sup_n \|x_n\|_{\ell^1} < \infty$ and $x_n(j) \to 0$ for all $j \geq 1$.