Homework 1

1. Show that in a Hilbert space H that $x \perp y$ if and only if $\|x - \alpha y\|_H = \|x + \alpha y\|_H$ for all scalars α.

2. Let H be a Hilbert space and suppose that $f, g \in H$ are linearly independent with $\|f\| = \|g\| = 1$. Show that for all $0 < t < 1$ that $\|tf + (1-t)g\| < 1$.

3. Let K be a non-empty linear subspace of a Hilbert space H. Show that K is dense in H if and only if $K^\perp = \{0\}$.

4. For the Hilbert space $H = \ell^2(\mathbb{N} \cup \{0\})$,

   (a) If $|\lambda| < 1$, and $L_\lambda h = \sum_{n=0}^{\infty} \lambda^n h_n$ for all $h \in H$, find the vector $\tilde{h} \in H$ such that $Lh = \langle h, \tilde{h} \rangle$ for all $h \in H$.

   (b) What is the norm of the linear functional defined in part (b)?

5. Suppose that H is a Hilbert space over the real numbers $\mathbb{R}$. Show that $\|x\| = \|y\|$ implies that

$$\langle x + y, x - y \rangle = 0.$$

6. Show that in an inner product space H that

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{1}{2}(x + y)\right\|^2.$$

This is called Apollonius' Identity.

7. If in a Hilbert space we have $x_n \to x$ and $y_n \to y$, show that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle .$$

8. Show that $y \perp x_n$ and $x_n \to x$ implies that $x \perp y$.

9. Show that for a sequence $\{x_n\}$ in a Hilbert space that the conditions $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ implies that $x_n \to x$.

10. Let $\{e_k\}$ be any orthonormal sequence in a Hilbert space H. Show that for any $x, y \in H$ that

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \leq \|x\| \, \|y\|$$

11. Show that an element x of a Hilbert space can not have "too many" Fourier coefficients $\langle x, e_k \rangle$ that are "big". Namely if $\{e_k\}$ is an orthonormal sequence, show that the number $N(\epsilon)$ of $\langle x, e_k \rangle$ such that

$$|\langle x, e_k \rangle| > \epsilon$$

must satisfy $N(\epsilon) < \frac{\|x\|^2}{\epsilon^2}$.