The Game of Parts

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Abstract

A two-player game with a simple strategy is an introduction to parity and proof by strong induction.

1 The Rules

Parts is a game for two players. The first one draws a circle with some tick-marks, or *darts*, pointing inside, as shown in the left of Figure 1. The second one makes the initial move, connecting the tips of two darts inside the circle with a curve, and then adding a *double dart* to the curve, as shown in the center of Figure 1. The first player then moves, connecting the tips of any two unused darts, as shown in the right of Figure 1. Curves must stay within the circle, and must not cross. The players alternate until there are no more moves. The last player to move wins.

2 The Game

Players pair up and play until someone discovers or conjectures a winning strategy. The single parameter in the game, the number N of darts, determines who wins regardless of how either player moves. In fact, only the parity of N matters, namely whether N is odd or even. One way to see this is to play Parts with ten initial darts for a while, and then play with nine initial darts. Another way is to play Winner's Parts, where Player 1 chooses N, but then the winner of a game becomes Player 1 for the next game. The player in a pair that discovers the winning strategy will be able to win consistently thereafter.

After someone finds a winning strategy, it should be formulated as precisely as possible, for example in the sentence "Player 1 will always win with nine initial darts" or "If $N = 11$, then Player 1 will always win."

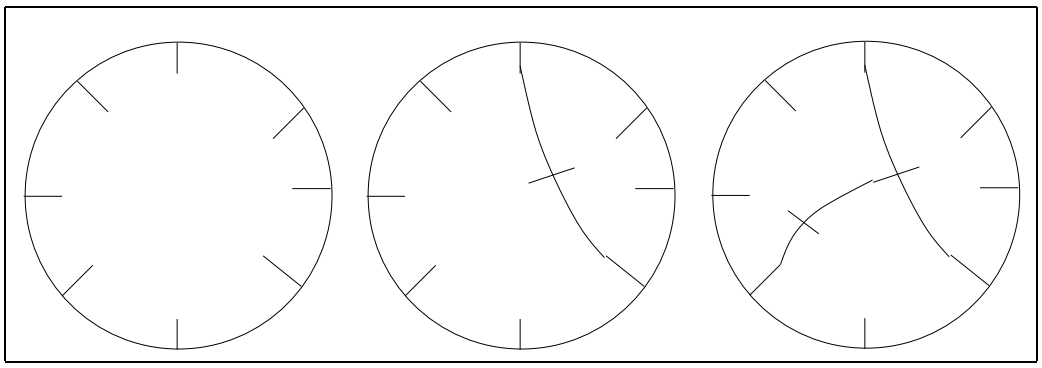


Figure 1: Left: circle with 8 darts, drawn by Player 1. Center: Player 2's first move, a curve with double dart. Right: Player 1's response.

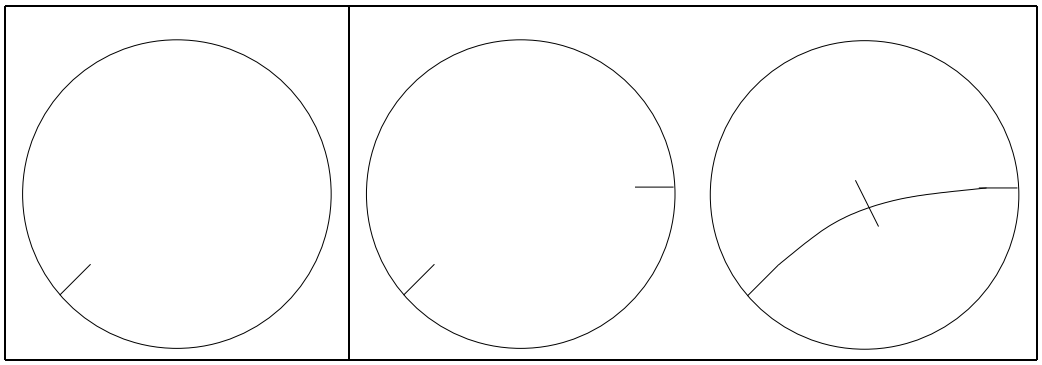


Figure 2: Left: $N = 1$ case, where Player 1 wins because Player 2 has no first move. Center: $N = 2$ case, where Player 2 wins. Right: Player 2's winning move in the $N = 2$ case.

3 The Analysis

In fact, Player 1 will always win if N is odd, while Player 2 will always win if N is even, regardless of how the curves are chosen. This translates into a winning strategy for Player 1: draw a circle with an odd number of darts.

To begin the analysis, consider the 1-dart and 2-dart cases in Figure 2. In the first, Player 1 wins immediately. In the second, Player 2 wins after making the only legal move.

When a player draws a curve between two darts, the part containing the darts is divided into two pieces. The curve's double dart replaces the two joined by the curve and used up, so the total number of darts remains the same, but they are divided among more parts. In other words, given a part with $N > 0$ darts, any move will result in two parts whose number of darts A and B will satisfy

$$N = A + B; \qquad 0 < A < N; \qquad 0 < B < N.$$

Thus the two subparts are each simpler, in the sense of having fewer darts, than the original part.

Now let $m(N)$ be the total number of moves that can be made starting from N darts. It is not clear that $m(N)$ is *well-defined*, since in our current ignorance we are not sure that there are not two ways to play the game starting from some N, with different total numbers of moves. But we are sure that $m(1) = 0$ and $m(2) = 1$, since we have checked those simple cases thoroughly. We conjecture:

Theorem 1 *If there are N darts in a part, then there are exactly $m(N) = N - 1$ moves in that part.*

Proof: The theorem is true for the cases $N = 1$ and $N = 2$, as we checked.

Now suppose that for some $N > 2$ we have checked all the cases with fewer than N darts. Any move in a part with N darts[1] will produce two subparts. If we write A and B for the number of darts in each subpart, then $N = A + B$ with $0 < A < N$ and $0 < B < N$. By our previous analysis, there are $m(A) = A - 1$ moves in the A subpart and $m(B) = B - 1$ moves in the B subpart, for a total of $m(A) + m(B) = A + B - 2$ moves. But one move was made to get two subparts, so from this number N initial darts there will always be a fixed number of total moves:

$$m(N) = 1 + m(A) + m(B) = 1 + (A - 1) + (B - 1) = A + B - 1 = N - 1.$$

Hence the total number of moves from N darts does not depend on how the game is played, so $m(N)$ is well-defined for all positive integers N, and we have shown that its formula is $m(N) = N - 1$. □

This proof uses the *Principle of Mathematical Induction*. Having checked $N = 1$ and $N = 2$, we can imagine how to show that $m(3) = 2$: the only way to write 3 as a sum of parts is $3 = 1 + 2 = 2 + 1$, and both cases 1 and 2 have been checked. That in turn implies $m(4) = 3$, since $4 = 1 + 3 = 2 + 2$, and so on.

Mathematical induction is a method of proving that a formula holds for infinitely-many numbers N, without having to consider all the numbers at once. It uses the idea that any single number N can be reached in a finite number of steps, but then lets us imagine the steps without actually performing them. We must check the first few cases, such as $N = 1$ and $N = 2$. This is called the *inductive anchor*. Then

[1] How many such initial moves are there?

we may assume that there is some least number N which has not yet been checked. We then derive the formula for this N from the formula known to hold at all smaller numbers. That is called the *inductive step*.

The number N is used as a variable. The theorem to be proved is a precisely-stated fact about N. The inductive anchor in the Parts theorem involved drawing a few pictures and enumerating the moves in the first two cases. The inductive step in the proof is made through some algebra with an identity, $N = A + B$.

4 Variations

How does the winning strategy change if a single rule is modified as follows, with all other rules staying the same:

1. *Two double darts are drawn at each move?* In that case, the total number of darts increases with each move, and the game never ends. Try to construct an example of an unending game with 2 initial darts.
2. *Double darts are used on the original circle, and curves may be drawn on the outside?* The outside of the circle is another part, like the inside. Starting with N double darts is therefore the same as playing two ordinary games of Parts with N darts each. If N is odd, Player 1 will have the last move in one game, so Player 2 will have to make the first move in the second game, and thus lose since N is odd. If N is even, Player 2 will start and win the first game, so Player 1 will start and win the second game. In both cases, Player 1 will always win.
3. *A single dart is drawn on one side of the curve at each move, with the player making the move choosing the side?* Here the players' choices influence the outcome. Try to construct an example with $N = 3$ in which there are either one or two legal moves.

5 Further Reading

Parts was inspired by the game of Sprouts, `http://www.madras.fife.sch.uk/maths/games/sprouts.html`, which has a more difficult mathematical analysis. A discussion of that game and its variants may be found in Martin Gardner's book:

1 Martin Gardner. *Mathematical Carnival*. The Mathematical Association of America, Washington, DC, 1988. ISBN 0-88385-448-1. URL `http://www.maa.org/pub/books/mcr.html`

Many more examples of proof by mathematical induction, as well as other techniques of mathematical analysis and proof, may be found in the excellent textbook by Steven Krantz:

2 Steven G. Krantz. *Techniques of Problem Solving*. American Mathematical Society, Providence, Rhode Island, 1997. ISBN 0-8218-0619-X. URL `http://www.ams.org/bookstore/rectexts`.

George Pólya's classic book on applications of mathematical reasoning to everyday situations also explains mathematical induction, as well as many ingenious methods of simplifying apparently complicated problems:

3 George Pólya. *How to Solve It*. Princeton University Press, Princeton, New Jersey, 1971. ISBN 0-691-02356-5. URL `http://pup.princeton.edu/titles/669.html`.