Singularity Detection in Images Using Dual Local Autocovariance*

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Introduction

Goal: Detect and characterize singularities in images and other functions by computing local dimension of the range.

Idea: Model images and functions as random vectors with Morrey–Campanato regularity. Use an autocovariance matrix as a substitute for the Jacobian, or derivative.

Application: Detect points (local dimension 0) and edges (local dimension 1) in images using the eigenvalues of the autocovariance matrix.

Prior Similar Ideas

Old Observation: if a function is rough along some direction, then its local Fourier transform stays big along that direction.

Duda and Hart [1972]: Hough transform for detection of lines and curves in images.

Mallat and Zhong [1992]: edge detection from wavelet maxima.

Aron and Kurz [1997]: linear hypothesis testing of variances in small windows to detect lines and edges.

New View: recognize nonsmooth directions from eigenvalues of an autocovariance matrix accumulated from the localized Fourier transform. In the regular case, the eigenvector of the larger eigenvalue will be normal to the edge.

Probabilistic Motivation (2-D)

Function $\phi : \mathbf{T} \to \mathbf{C}$ on the unit circle $\mathbf{T} \subset \mathbf{R}^2$.

Random direction vector $\Phi : \mathbf{T} \to \mathbf{C}^2$ is

$$\Phi(\theta) \stackrel{\text{def}}{=} \phi(\theta)(\cos\theta, \sin\theta), \qquad \theta \in \mathbf{T}.$$

Direction probability density is $|\phi|^2$, normalized.

Symmetric 2×2 Autocovariance Matrix

$$E(\Phi^*\Phi)_{ij} = \int_{\mathbf{T}} \Phi_i^* \Phi_j = \int_0^{2\pi} |\phi(\theta)|^2 \tau_i(\theta)\tau_j(\theta)\, d\theta,$$

for $i, j \in \{1, 2\}$, $\tau_1(\theta) = \sin\theta$, and $\tau_2(\theta) = \cos\theta$.

Then $\langle E(\Phi^*\Phi)v, v\rangle = \int_{\mathbf{T}} |\langle \Phi, v\rangle|^2$

– attain $\sup_{\|v\|=1} \langle E(\Phi^*\Phi)v, v\rangle$ at a unit eigenvector $v \in \mathbf{R}^2$ of the largest eigenvalue of $E(\Phi^*\Phi)$.

– Such an eigenvector v always exists.

– Use v to approximate the max of $|\phi| = \|\Phi\|$.

Example (2-D)

For $f : \mathbf{R}^2 \to \mathbf{C}$ and large fixed $R > 0$, define

$$\phi(\theta) = |\widehat{f}(R\cos\theta, R\sin\theta)|^2$$

This samples $|\widehat{f}|^2$ far out in the direction θ.

Suppose ϕ is highly concentrated near the point $\theta_0 \in \mathbf{T}$. Then $E(\Phi^*\Phi)$ is approximately proportional to

$$\begin{pmatrix} \cos^2\theta_0 & \cos\theta_0 \sin\theta_0 \\ \cos\theta_0 \sin\theta_0 & \sin^2\theta_0 \end{pmatrix}.$$

Any nonzero vector in the θ_0 direction is an eigenvector of the largest eigenvalue.

Alternatives

(A-1) Replace circle $\mathbf{T}$ with disc B:

$$E(\Phi^*\Phi)_{ij} = \int_B |\phi(\xi)|^2 \xi_i \xi_j \, d\xi,$$

for $i, j \in \{1, 2\}$, and $\phi \in L^2(B)$.

(A-2) Replace disk B with $\mathbf{R}^2$, under additional integrability assumptions.

Localization

Start with $f : \mathbf{R}^2 \to \mathbf{R}$, point of interest x, scale $\epsilon = 1/R$.

Localize by $f \mapsto gf$, for some nonzero radial Schwartz function $g : \mathbf{R}^2 \to \mathbf{R}$:

$$g_\epsilon(y) = g\left(\frac{y-x}{\epsilon}\right), \qquad \epsilon > 0.$$

which is:

- smooth, to avoid introducing new singularities;
- radial, to avoid introducing directional bias;
- nonzero and concentrated, to emphasize the point of interest x.

Dual Local Autocovariance

Definition 1 *The* dual local autocovariance matrix of f at x *is the* 2×2 *matrix whose* ij*-coefficient is:*

$$E_{\epsilon,g}(f;x)_{ij} = \int_{B(0,1/\epsilon)} \xi_i \xi_j \left|\widehat{g_\epsilon f}(\xi)\right|^2 \, d\xi.$$

This is the real, symmetric second moment matrix of the unnormalized probability density function $|\widehat{g_\epsilon f}|^2$.

Localization gf is integrable $\Rightarrow$ $\widehat{gf}$ is bounded and continuous $\Rightarrow$ matrix coefficients are well defined.

$f \not\equiv 0$ near $x \Rightarrow E_{\epsilon,g}(f;x)$ is positive definite.

Straight Edges

Change of variables: $E_{\epsilon,g}(f;x)_{ij} =$

$$\int_B \xi_i \xi_j \left| \int_{\mathbf{R}^2} g\left(y + \frac{\epsilon - 1}{\epsilon} x\right) f(\epsilon y) e^{-2\pi i y \cdot \xi}\, dy \right|^2 d\xi.$$

If f is homogeneous of degree 0, and $x = 0$, this formula is independent of $\epsilon > 0$

Example: $f = \mathbf{1}_L$, the characteristic function of the left half-plane $L = \{(x_1, x_2) : x_1 \leq 0\}$.

– $E_{\epsilon,g}(\mathbf{1}_L; 0)$ is a matrix with no ϵ-dependence.

– Use $g(x) = \exp(-\pi|x|^2)$, for ease of computing Fourier transforms.

– Get dual local autocovariance matrix explicitly for the edge $x_1 = 0$.

Straight edges (continued. . .)

Lemma 1 $E_{\epsilon,g}(1_L; 0)$ *has the eigensystem*

$$\lambda_1 \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda_2 \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

satisfying $\lambda_1 > \lambda_2 > 0$, *independently of* ϵ.

Numerical estimate: the matrix is diagonal, so

$$\lambda_i = \int_B \xi_i^2 \left| e^{-\pi\xi_2^2} \int_{-\infty}^{0} e^{-\pi x^2 - 2\pi i x \xi_1}\, dx \right|^2 d\xi,$$

for $i = 1, 2$. In terms of the Gaussian integral $G(\xi) = \int_0^\xi \exp(\pi x^2)\, dx$, this becomes:

$$\lambda_i = \int_B \xi_i^2 e^{-2\pi|\xi|^2} \left[\frac{1}{4} + G^2(\xi_1) \right] d\xi,$$

for $i = 1, 2$.

By *Mathematica*, $\lambda_1 - \lambda_2 \approx 0.02258364$.

Off the Straight Edge

Conversely, by taking $g(\xi)$ to be a smooth radial function supported in $B(0,1)$, we see:

– for any $x \notin L$, and $0 < \epsilon < \text{dist}\,(L, x)$ small enough,

$$E_{\epsilon,g}(\mathbf{1}_L; x) \equiv 0,$$

since $g_\epsilon \mathbf{1}_L = 0$.

– for each point $x \in \text{Int}L$,

$$E_{\epsilon,g}(\mathbf{1}_L; x) = const \cdot I,$$

for all sufficiently small $0 < \epsilon < \text{dist}\,(x, \partial L)$.

In either case, E has two equal eigenvalues.

Rotation and Translation

Translation moves x, preserves eigenvectors:

Lemma 2 *Let T be a translation by x in $\mathbf{R}^2$: $f \circ T(y) = f(y + x)$. Then*

$$E_{\epsilon,g}(f; x) = E_{\epsilon,g}(f \circ T; 0).$$

Rotation about x rotates eigenvectors, changes edge direction:

Lemma 3 *Let U be a rotation about x in $\mathbf{R}^2$. Then*

$$E_{\epsilon,g}(f \circ U; x) = U \circ E_{\epsilon,g}(f; x) \circ U^{-1}.$$

Straight Edge Detection

Let 1_H be the characteristic function of a half-plane

$$H = \{x \in \mathbf{R}^2 : \nu \cdot x \leq \beta\},$$

with nonzero normal vector $\nu \in \mathbf{R}^2$, for some constant $\beta \in \mathbf{R}$. Its *edge* is the line ∂H.

Theorem 4 *$x \in \partial H$ iff $E_{\epsilon,g}(1_H; x)$ has distinct eigenvalues $\lambda_1 > \lambda_2 > 0$ for every smooth, radial function $g \not\equiv 0$, and $\epsilon > 0$. In that case, ν will be an eigenvector of the larger eigenvalue.*

Note; degree-0 homogeneity of $1_H \Rightarrow$ eigenvalues of $E_{\epsilon,g}(1_H; x)$ do not depend on ϵ.

For $x \notin \partial H$ off the edge, and each smooth, compactly-supported radial function g,

$$E_{\epsilon,g}(1_H; x) = const \cdot I$$

has eventually constant and equal eigenvalues as $\epsilon \to 0$.

Smooth-Curve Edges

Idea: Boundaries of smooth domains look like boundaries of half-planes.

Fix:

– Domain $D \subset \mathbf{R}^2$, smooth boundary ∂D.

– $\mathbf{1}_D$, the characteristic function of D.

– $g : \mathbf{R}^2 \to \mathbf{R}$, a smooth radial function, supported in $B = B(0, 1)$.

– H, half-plane with $\partial H \parallel \partial D$ at some x.

As $\epsilon \to 0$, expect:

– $x \in \partial D \Rightarrow$ distinct eigenvalues for $E_{\epsilon,g}(\mathbf{1}_D, x)$.

– $x \notin \partial D \Rightarrow$ equal eigenvalues for $E_{\epsilon,g}(\mathbf{1}_D, x)$.

Smooth-Curve Edges (continued. . .)

Lemma 5 *In any matrix norm,*

$$\|E_{\epsilon,g}(\mathbf{1}_H; x) - E_{\epsilon,g}(\mathbf{1}_D; x)\| = O(\epsilon),$$

as $\epsilon \to 0$.

(Note: for Schwartz g, any $\delta > 0$ gives

$$\|E_{\epsilon,g}(\mathbf{1}_H; x) - E_{\epsilon,g}(\mathbf{1}_D; x)\| = O(\epsilon^{1-\delta}),$$

as $\epsilon \to 0$.)

Theorem 6 *For* $x \in \partial D$, *and eigenvalues* $\lambda_1(\epsilon)$ *and* $\lambda_2(\epsilon)$ *of* $E_{\epsilon,g}(\mathbf{1}_D; x)$, *we have*

$$\liminf_{\epsilon \to 0+} |\lambda_1(\epsilon) - \lambda_2(\epsilon)| > 0.$$

For $x \notin \partial D$, $|\lambda_1(\epsilon) - \lambda_2(\epsilon)| \to 0$ *as* $\epsilon \to 0$.

Morrey-Campanato Edges

Fix a domain $D \subset \mathbf{R}^2$ with boundary ∂D in the Morrey-Campanato class $L(\rho, \infty, 1)$, for fixed $\rho(\epsilon) = o(\epsilon)$ as $\epsilon \to 0+$. Also suppose:

– 1_D is the characteristic function of D.

– For $x \in \partial D$ and $\epsilon > 0$, there is a line ℓ_ϵ realizing $\inf \|\partial D - \ell_\epsilon\|_{L(\rho,\infty,1)}$ over $B(x, \epsilon)$.

– $g : \mathbf{R}^2 \to \mathbf{R}$ is a smooth radial function supported in $B = B(0, 1)$.

– $H = H_\epsilon$ is a half-plane with $\partial H_\epsilon = \ell_\epsilon$.

Theorem 7 *In any matrix norm,*

$$\lim_{\epsilon \to 0+} \|E_{\epsilon,g}(1_H; x) - E_{\epsilon,g}(1_D; x)\| = 0.$$

Hence, if f is discontinuous along a Morrey–Campanato curve, then at each point x on the curve, $E_{\epsilon,g}(f; x)$ has distinct eigenvalues for all sufficiently small $\epsilon > 0$.

Affine and Differentiable Functions

At points x where f is differentiable, the eigenvalues $\lambda_1(\epsilon), \lambda_2(\epsilon)$ of $E_{\epsilon,g}(f;x)$ are equal in the limit:

Lemma 8 *Let $A : R^2 \to \mathbf{R}$ be an affine function: $A(x) = a \cdot x + b$ for some constants $a \in \mathbf{R}^2, b \in \mathbf{R}$. Let $g : \mathbf{R}^2 \to \mathbf{R}$ be a smooth radial function supported in $B = B(0,1)$. Then:*

$$\lim_{\epsilon \to 0+} [\lambda_1(\epsilon) - \lambda_2(\epsilon)] = 0.$$

Lemma 9 *If $f : \mathbf{R}^2 \to \mathbf{R}$ is differentiable at x with affine A tangent to f at x, then*

$$\|E_{\epsilon,g}(f;x) - E_{\epsilon,g}(A;x)\| = o(\epsilon),$$

as $\epsilon \to 0$.

(Continuously differentiable f near x satisfies

$$\|E_{\epsilon,g}(f;x) - E_{\epsilon,g}(A;x)\| = O(\epsilon^2),$$

as $\epsilon \to 0$.)

Detecting Non-Edge Points

Theorem 10 *Suppose that* $f : \mathbf{R}^2 \to \mathbf{R}$ *is differentiable at* x*. Then for any smooth radial compactly-supported function* $g : \mathbf{R}^2 \to \mathbf{R}$*,*

$$E_{\epsilon,g}(f; x) \to const \cdot I,$$

as $\epsilon \to 0$*.*

Converse is false: $\lim_{\epsilon \to 0+} |E_{\epsilon,g}(f; x)| = const \cdot I$ does not imply that f is differentiable at x, or even continuous. Symmetry can masquerade as smoothness, as in $f = \mathbf{1}_{x_1,x_2>0}$. Fix $g(x) = \exp(-\pi|x|^2)$ as before. Then

$$E_{\epsilon,g}(f; 0) = const \cdot I \neq 0$$

for every $\epsilon > 0$, so $\lambda_1(\epsilon) = \lambda_2(\epsilon) = \lambda > 0$ for every ϵ, even though f is discontinuous at 0.

Higher-Dimensional Theory

Study the geometry of $f : \mathbf{R}^p \to \mathbf{R}^d$ in high dimensions p, d.

Suppose $f = (f_1, \ldots, f_d)$ is polynomially bounded. Fix $x \in \mathbf{R}^p$, radial Schwartz $g : \mathbf{R}^p \to \mathbf{R}^d$, and $g_\epsilon(y) = g\left(\frac{y-x}{\epsilon}\right)$, as before, for $\epsilon > 0$.

Definition 2 *The* dual local autocovariance matrix of f at x *is the* $p \times p$ *matrix:*

$$E_{\epsilon,g}(f; x)_{ij} = \int_{B(0,1/\epsilon)} \xi_i \xi_j \|\widehat{f \cdot g_\epsilon}(\xi)\|^2 \, d\xi,$$

for $i, j \in \{1, \ldots, p\}$.

Smooth $p \times p$ case is like smooth 2×2:

Theorem 11 *Suppose that* $f : \mathbf{R}^p \to \mathbf{R}^d$ *is differentiable at* x. *Then for any smooth radial function* $g : \mathbf{R}^p \to \mathbf{R}^d$ *of compact support, the matrix* $E_{\epsilon,g}(f; x) \to const \cdot I$, *as* $\epsilon \to 0$.

Morrey–Campanato Hypersurfaces

Suppose domain $D \subset \mathbf{R}^p$ has ∂D in the generalized Morrey–Campanato class $L(\rho, \infty, 1)$ for fixed $\rho(\epsilon) = o(\epsilon)$, as $\epsilon \to 0$.

Fix a smooth radial function g supported in $B(0,1) \subset \mathbf{R}^p$.

Theorem 12 *Fix $x \in \partial D$, and suppose that for every $\epsilon > 0$, there exists a hyperplane ℓ_ϵ that realizes* $\inf \|\partial D - \ell_\epsilon\|_{L(\rho,\infty,1)}$ *over $B(x,\epsilon)$. Let $H = H_\epsilon$ be a half-space with $\partial H_\epsilon = \ell_\epsilon$. Then*

$$\lim_{\epsilon \to 0} \|E_{\epsilon,g}(\mathbf{1}_H; x) - E_{\epsilon,g}(\mathbf{1}_D; x)\| = 0.$$

For $x \in \partial H$, the eigenvalues of $E_{\epsilon,g}(\mathbf{1}_H, x)$ may be numbered to satisfy:

$$\lambda_1(\epsilon) > \lambda_j(\epsilon) > 0$$

for all $j = 2, \ldots, p$, independent of ϵ.

Discretization

Discrete Fourier transform on N real samples $\{f(n) : 0 \leq n < N\}$:

$$\widehat{f}(k) = \sum_{n=0}^{N-1} \exp\left(-2\pi i \frac{kn}{N}\right) f(n),$$

for integer $k \in B_N = \left[-\frac{N}{2}, \frac{N}{2}\right]$.

If only the first $q \ll N$ samples of f are nonzero, then the sum is over $\{0, 1, \ldots, q-1\}$.

When f is real-valued,

$$|\widehat{f}(k)|^2 = \sum_{n,n'=0}^{q-1} \exp\left(-2\pi i \frac{k(n-n')}{N}\right) f(n) f(n'),$$

for $k \in B_N$.

Discretization (continued. . .)

The r-th moment of $|\widehat{f}(k)|^2$ is $\sum_{k\in B_N} k^r |\widehat{f}(k)|^2 =$

$$\sum_{n,n'=0}^{q-1} f(n)f(n') \sum_{k\in B_N} k^r \exp\left(-2\pi i \frac{k(n-n')}{N}\right).$$

The innermost sum in k, if normalized with N^{r+1}, is a Riemann sum for the integral

$$\mu_r(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^r \exp(-2\pi i n x)\, dx,$$

evaluated at $n \leftarrow (n-n')$:

$$\mu_0(n) = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mu_1(n) = \begin{cases} 0, & \text{if } n = 0, \\ i\frac{(-1)^n}{2\pi n}, & \text{otherwise;} \end{cases}$$

$$\mu_2(n) = \begin{cases} 1/12, & \text{if } n = 0, \\ \frac{(-1)^n}{2\pi^2 n^2}, & \text{otherwise.} \end{cases}$$

Discrete Dual Local Autocovariance

Image: f on $\{(m,n) : 0 \leq m < M;\ 0 \leq n < N\}$.

Bump: g on $\{(m,n) : 0 \leq m < p; 0 \leq n < q\}$, for $p \ll M; q \ll N$.

Autocovariance matrix: localize $f \leftarrow gf$, then compute

$$E_{11} = \sum_{m,m'=0}^{p-1} \sum_{n=0}^{q-1} f(m,n) f(m',n) \mu_2(m-m');$$

$$E_{22} = \sum_{m=0}^{p-1} \sum_{n,n'=0}^{q-1} f(m,n) f(m,n') \mu_2(n-n');$$

$$E_{12} = E_{21} =$$

$$\sum_{m,m'=0}^{p-1} \sum_{n,n'=0}^{q-1} f(m,n) f(m',n') \mu_1(m-m') \mu_1(n-n').$$

Implementation

Around each grid point x of the image, do:

Localization. Extract the samples on the square subgrid $x+[-\epsilon,\epsilon]^2$. Multiply $f(x+y)$ by Gaussian bump $g(y) = \exp(-\pi|y|^2/\epsilon^2)$.

Dual autocovariance. Compute $E = (E_{ij})$, a real-valued, symmetric, positive semidefinite 2×2 matrix.

Eigenvalues. For symmetric 2×2 matrices E:

$$\lambda = \frac{1}{2}\left(E_{11} + E_{22} \pm \sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}\right),$$

where $+$ gives λ_1, $-$ gives λ_2.

Edginess. Compute ratio λ_1/λ_2: greater relative difference $\Rightarrow$ greater edginess. Or use reciprocal $\lambda_2/\lambda_1 \in [0,1]$ in write-black mode: darker marks $\Rightarrow$ greater edginess.

Example Images

1. Geometrical figures, piecewise constant functions with jump discontinuities along rectifiable, mostly smooth curves.

2. Fingerprint, part of NIST's compliance test suite for the FBI's WSQ compression standard.

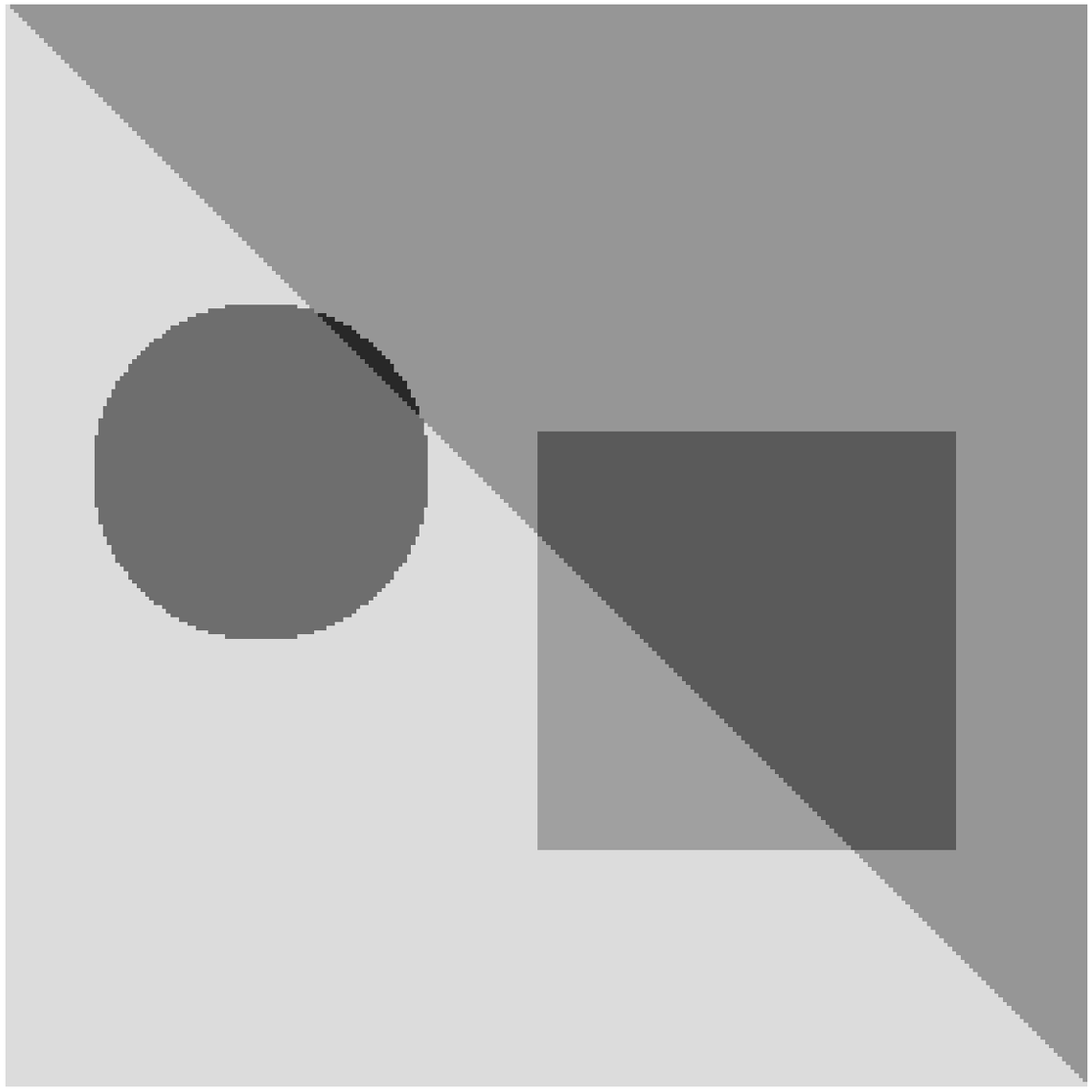
3. Lena Sjøblum, the famous woman in a hat.

4. Cone, a ray-traced image by Craig Kolb.

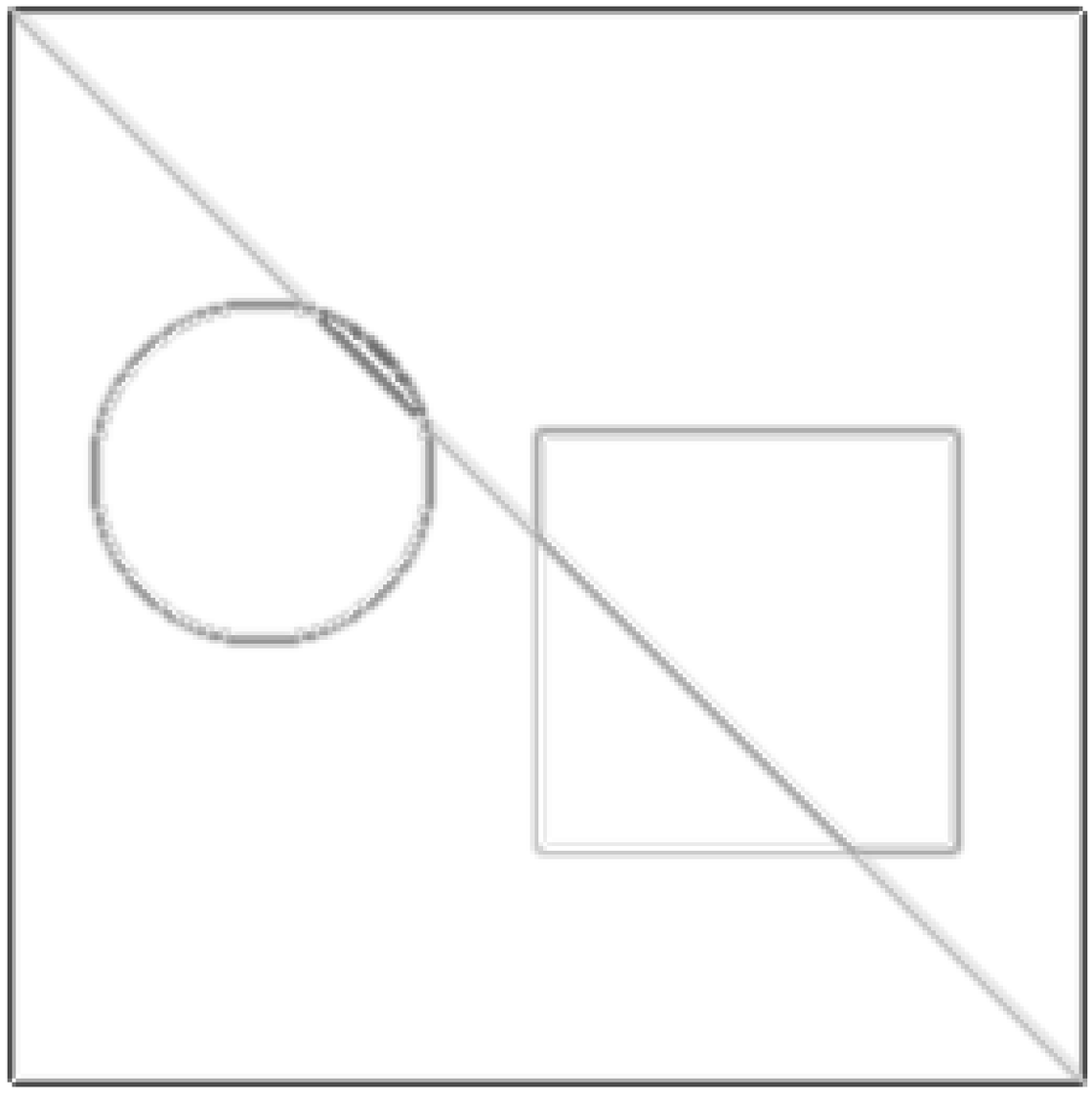
5. Truck, one frame of video from Peng Li.

Use $\epsilon = 3$, $\Rightarrow$ localize to 7×7 subgrids.

Example: Geometrical Figures



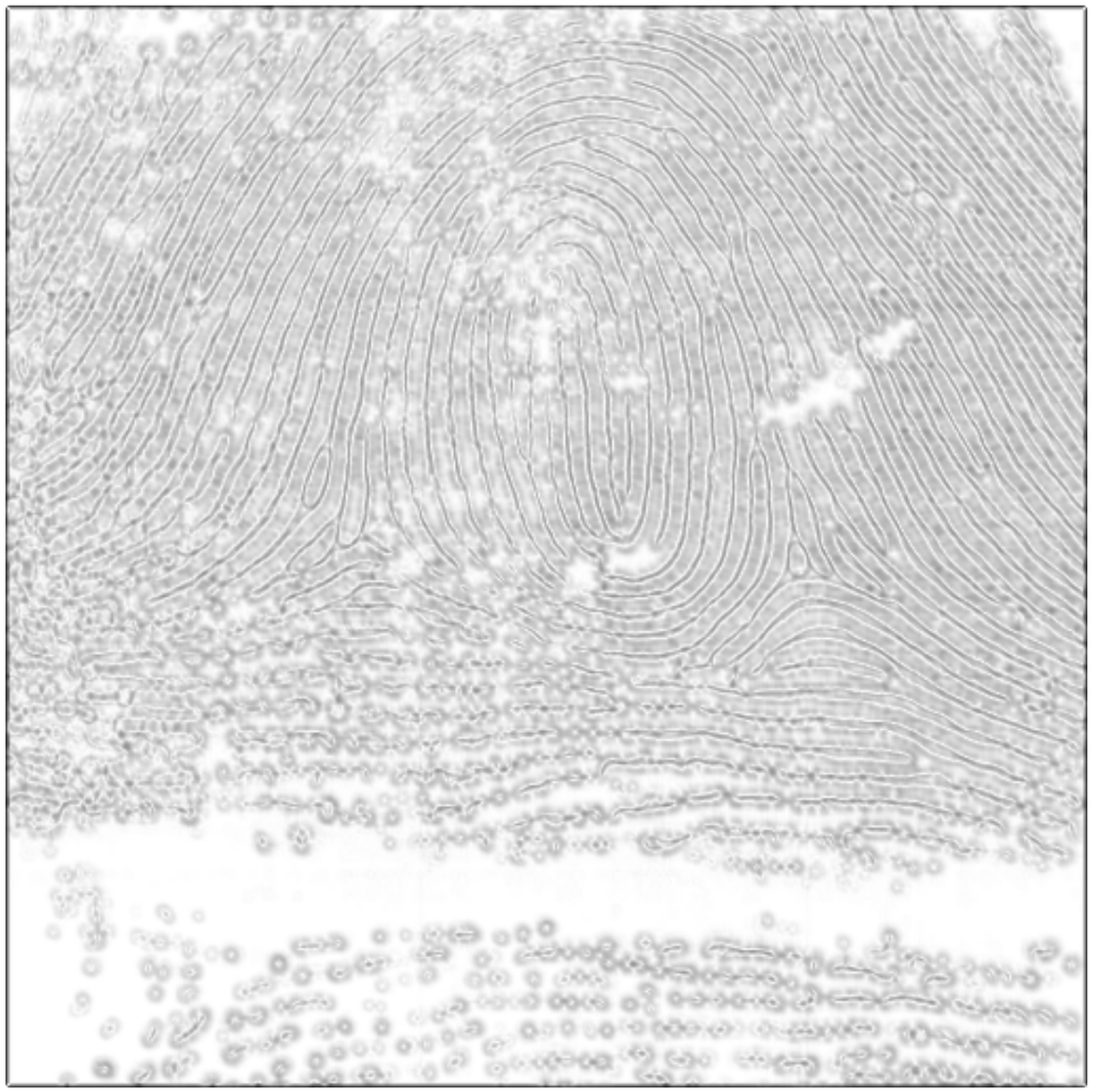
Geometrical Figures Edginess



Example: Fingerprint Image



Fingerprint Edginess



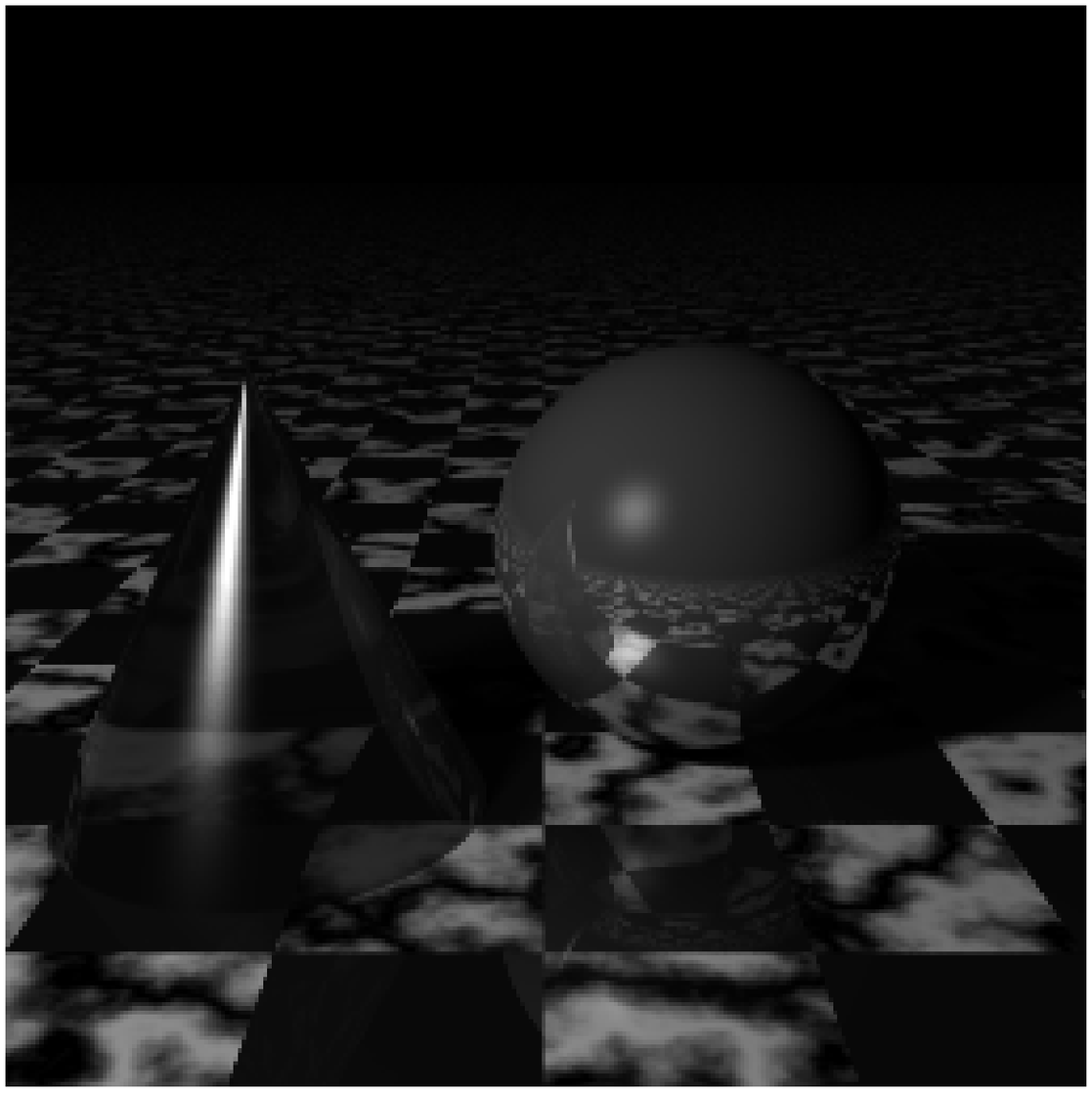
Example: Lena Sjøblum Image



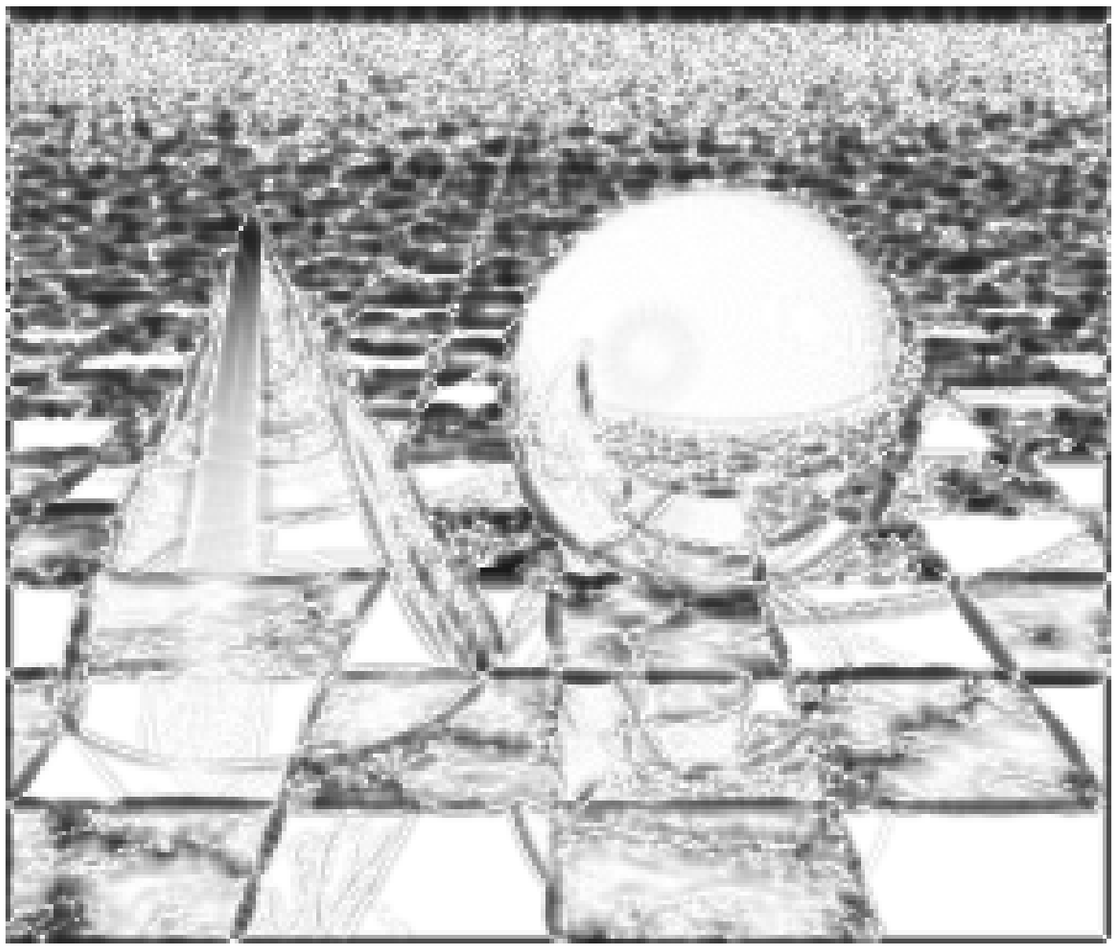
Lena Edginess



Example: Cone Image



Cone Edginess



Example: Truck Video Frame



Truck Edginess

