Some Theory,
But More Applications
of Wavelet Packets

M. Victor Wickerhauser

Department of Mathematics
Washington University
St. Louis, Missouri, 63130 USA

victor@math.wustl.edu
www.math.wustl.edu/~victor

University of Central Oklahoma
October 25, 2002

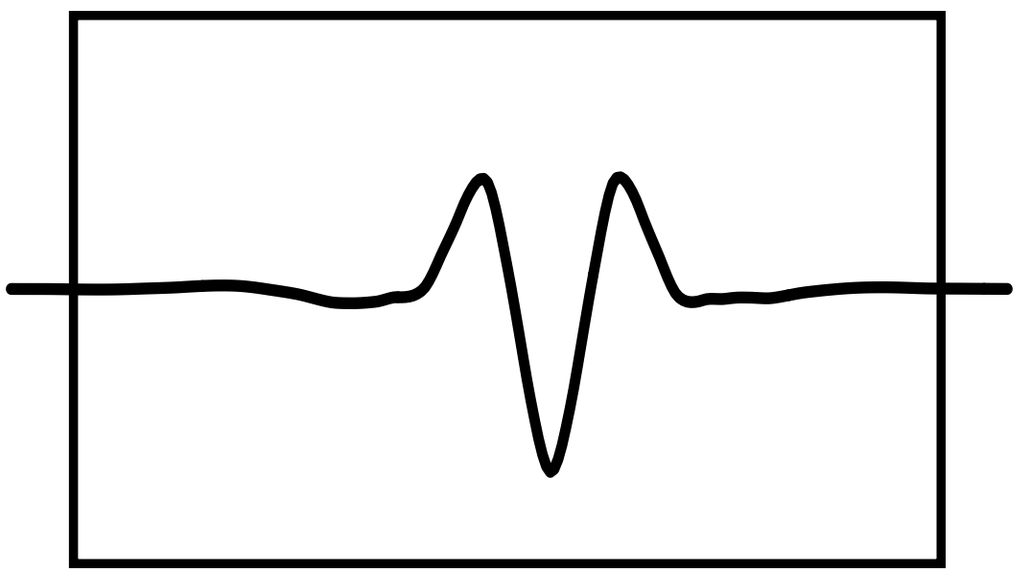
Definition

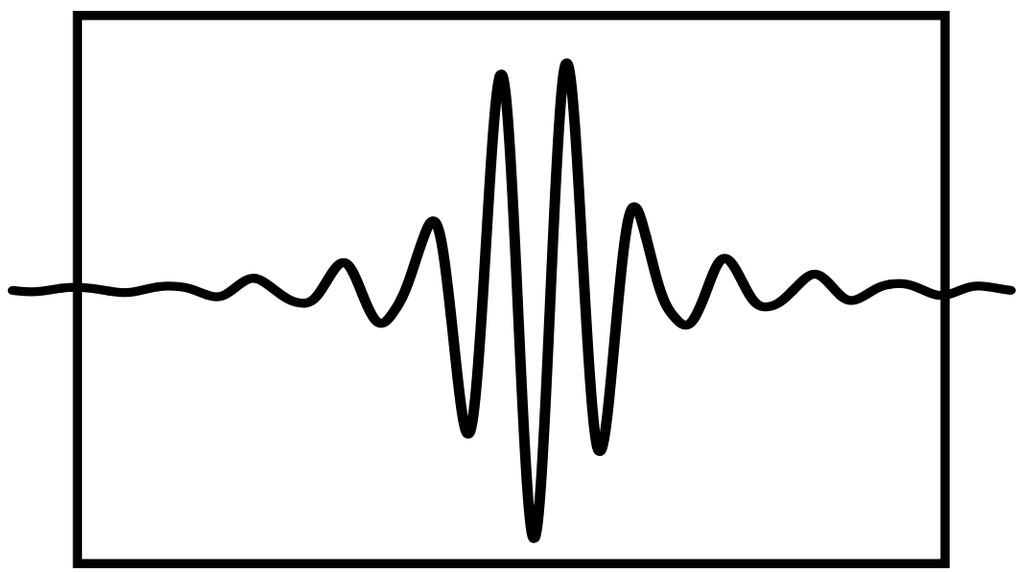
A *Wavelet* $w = w(t)$ is a nice function which is

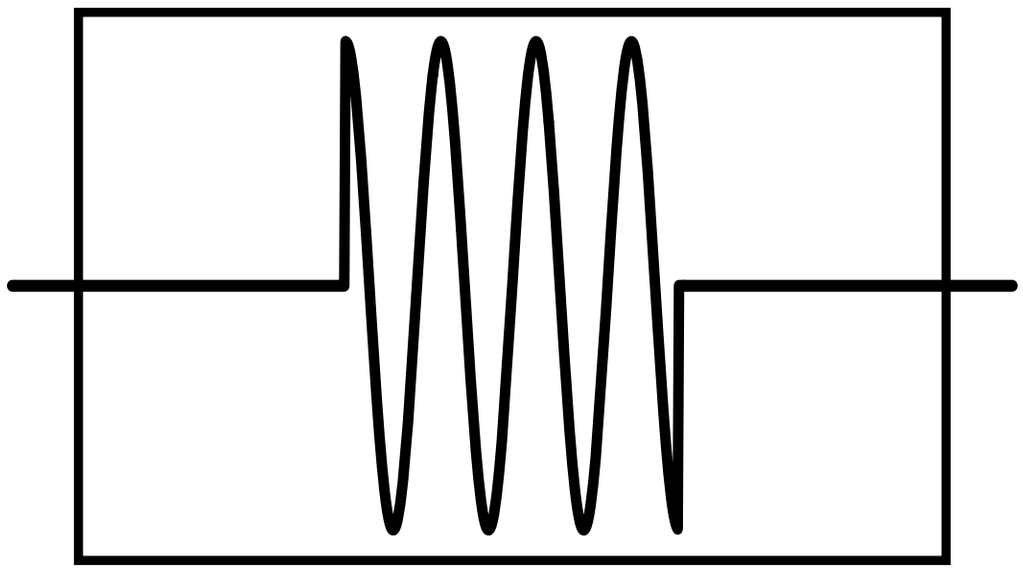
1. localized in time
2. localized in frequency

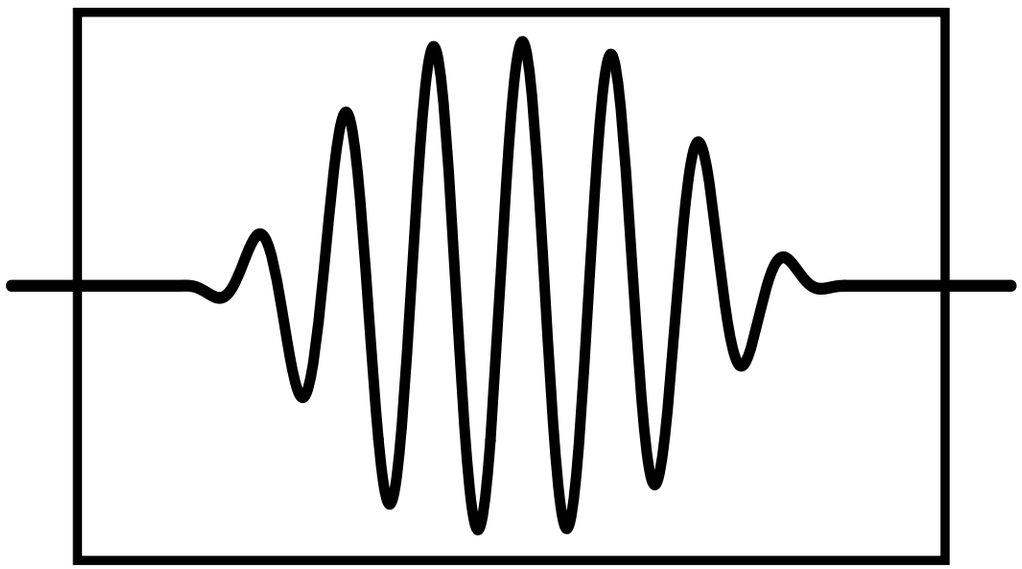
and which can be superposed, together with copies of itself produced by transformations like shifts, dilations, or modulations, to produce any desired finite-energy signal.

Example waveforms









History

- Fourier bases (1822, Paris)
- Haar bases (1910, *Math. Annalen*)
- Gabor functions (1946, *J. IEE*)
- Balian-Low theorem (1981, *CRAS*)
- Wilson bases (1987, Cornell)
- Compactly-supported smooth ortho-normal wavelets (1988, *CPAM*)
- Malvar "LOT" (1990, *IEEE ASSP*)
- Biorthogonal wavelets, wavelet packets, best basis, denoising (1992, *IEEE IT*)
- WSQ fingerprint standard (1993, FBI)
- Local discriminant bases (1994, *CRAS*)
- Multiwavelets (1994, *OE*)
- Wavelets on spheres (1995, *ACM*)
- Sweldens "lifting" (1996, *ACHA*)
- Ridgelets, edgelets, brushlets; spatio-temporal, non-stationary, tight-frame wavelets,...
- JPEG-2000 compression (1999)

Variations on $w_{ab}(t) = w(at + b)$

- continuous indices $a > 0$, b
- discrete indices $a = 2^{-j}$, $b \in \mathbf{Z}$
- orthonormal $\{w_{jk}\}$
- biorthogonal $\{w_{jk}\}, \{w'_{jk}\}$
- symmetric, antisymmetric
- multidimensional
- matrix dilations a
- other parameters $\{w_{abc...}\}$
  - frequency
  - rotation angle
- discrete or finite domain x
- adjusted to intervals
- adjusted to curved manifolds
- multiple filters

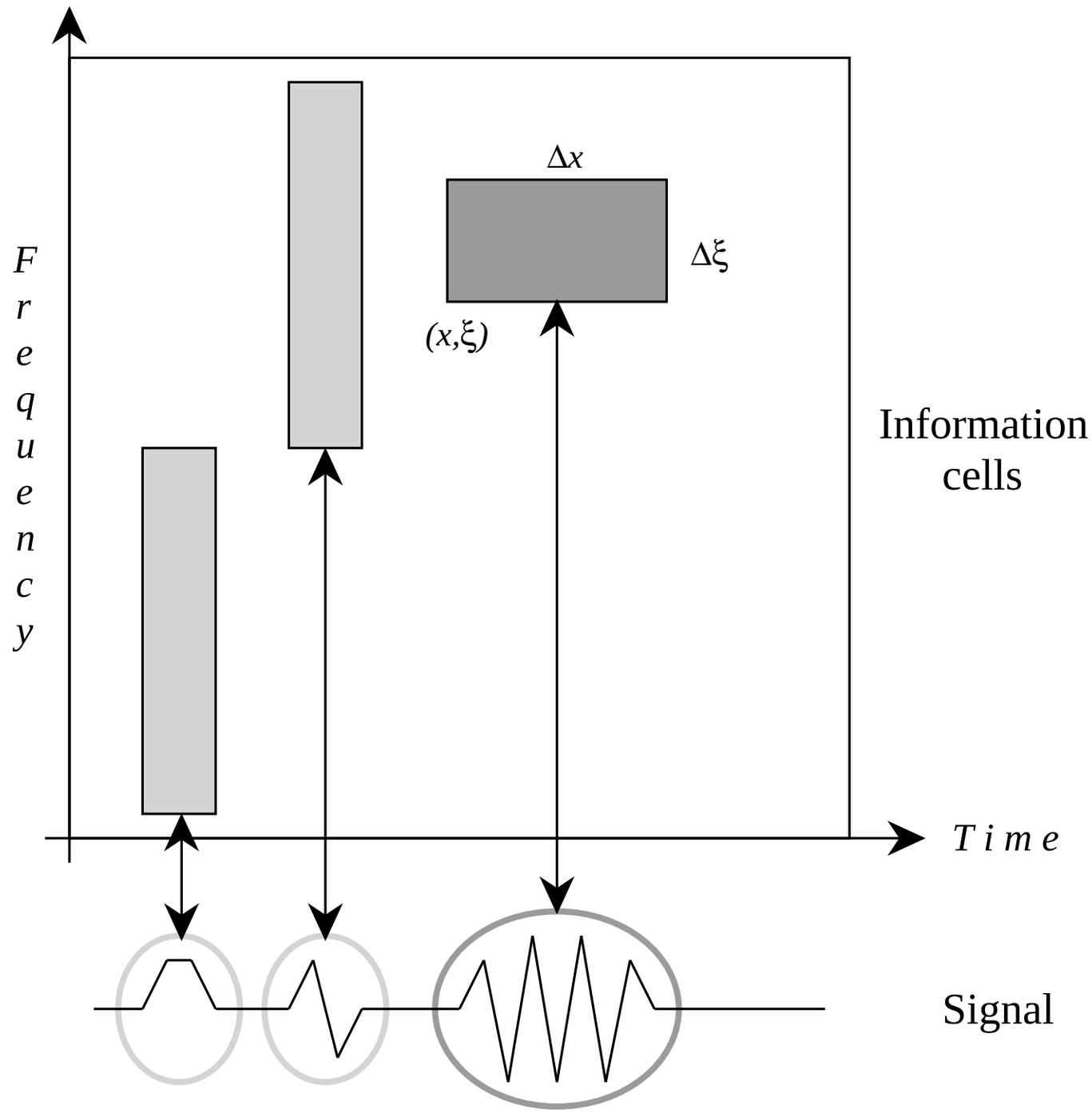
Difficulties and Solutions

- Few transform standards
  - + indexing conventions
  - + consistent definitions
  - + uniform nomenclature
- Little evaluation beyond small trials
  - + use NIST, TIMIT, etc.
  - + trade secrets, proprietary information, and patents
  - + competition with highly engineered prior art
- Strong mathematical preparation needed
  - + new undergraduate courses
  - + new graduate programs

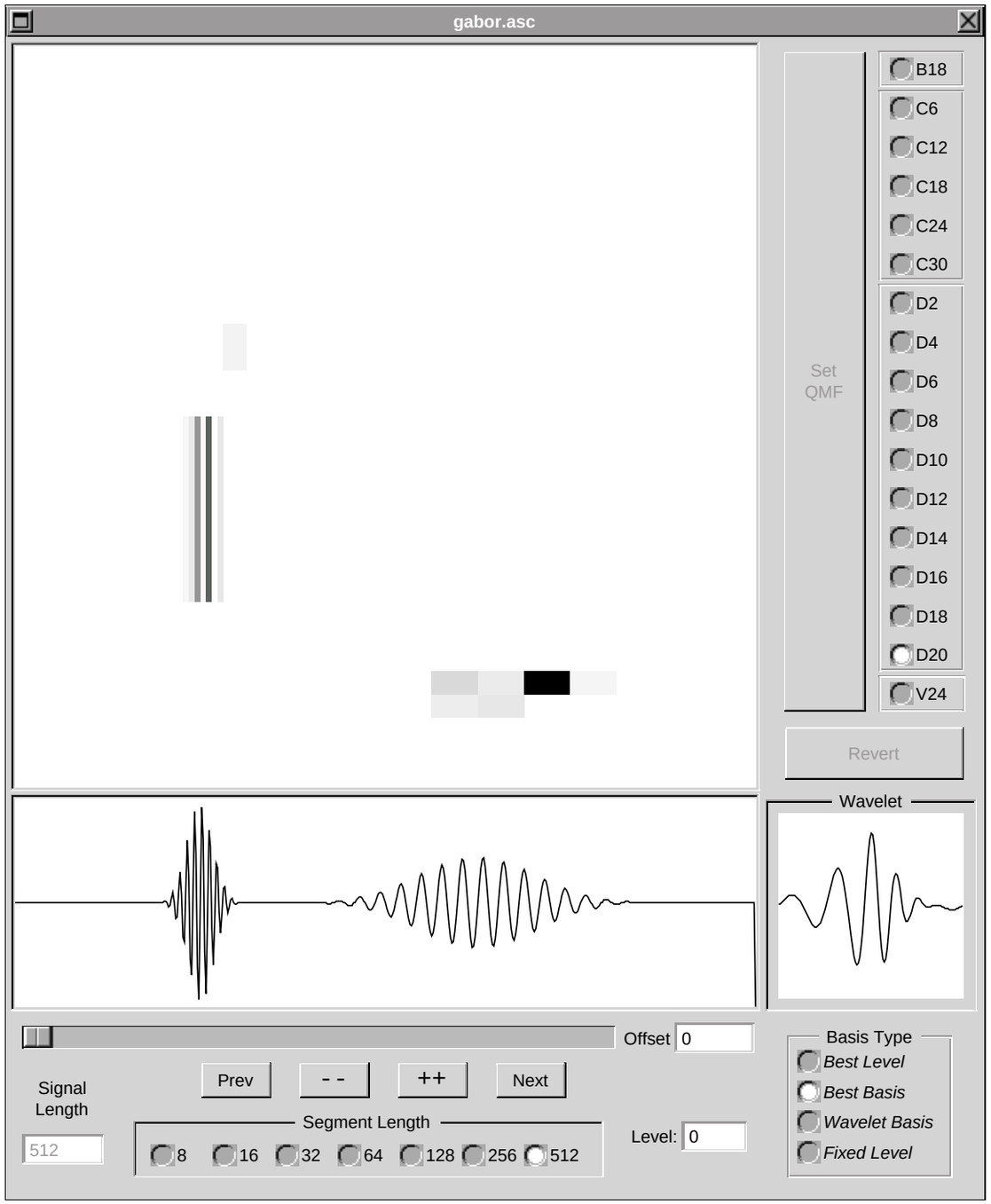
Literature

- Bibliographies:
  - Pittner, *et al.*, *Wavelet Literature Survey*, TU-Vienna 1993. [∼1000 titles]
  - `http://www.wavelet.org`, Wavelet Digest email list. [∼17,000 subscribers]
- Journals:
  - Appl. Comp. Harm. Anal.
  - J. Fourier Anal. Appl.
  - SIAM J. Math. Anal.; Num. Anal.
  - IEEE SP; IT; ...
  - SPIE Optical Eng.
  - Comm. Pure Appl. Math.
  - J. Math. Physics
  - Digital Signal Processing
  - Dr Dobb's Journal

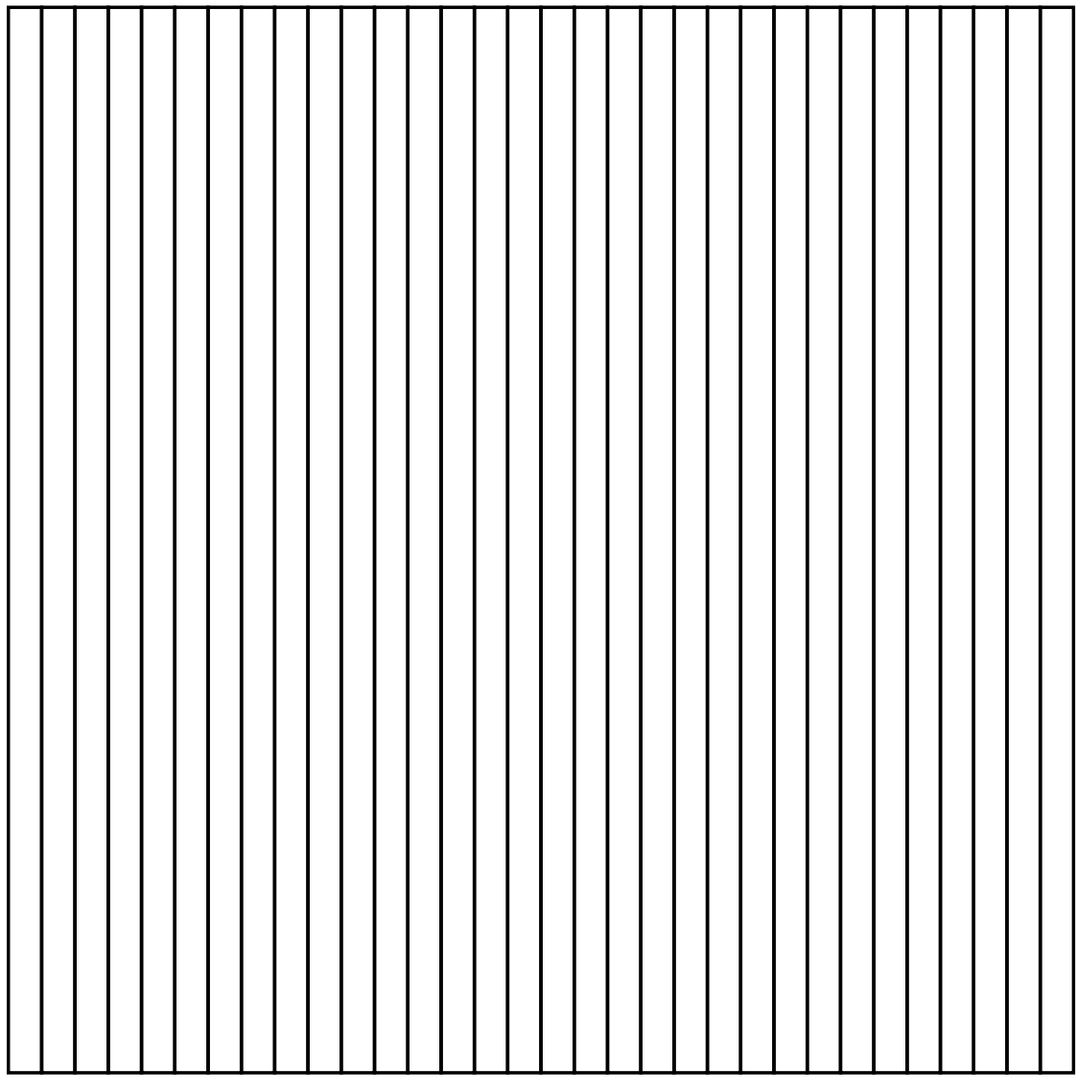
Information Cells

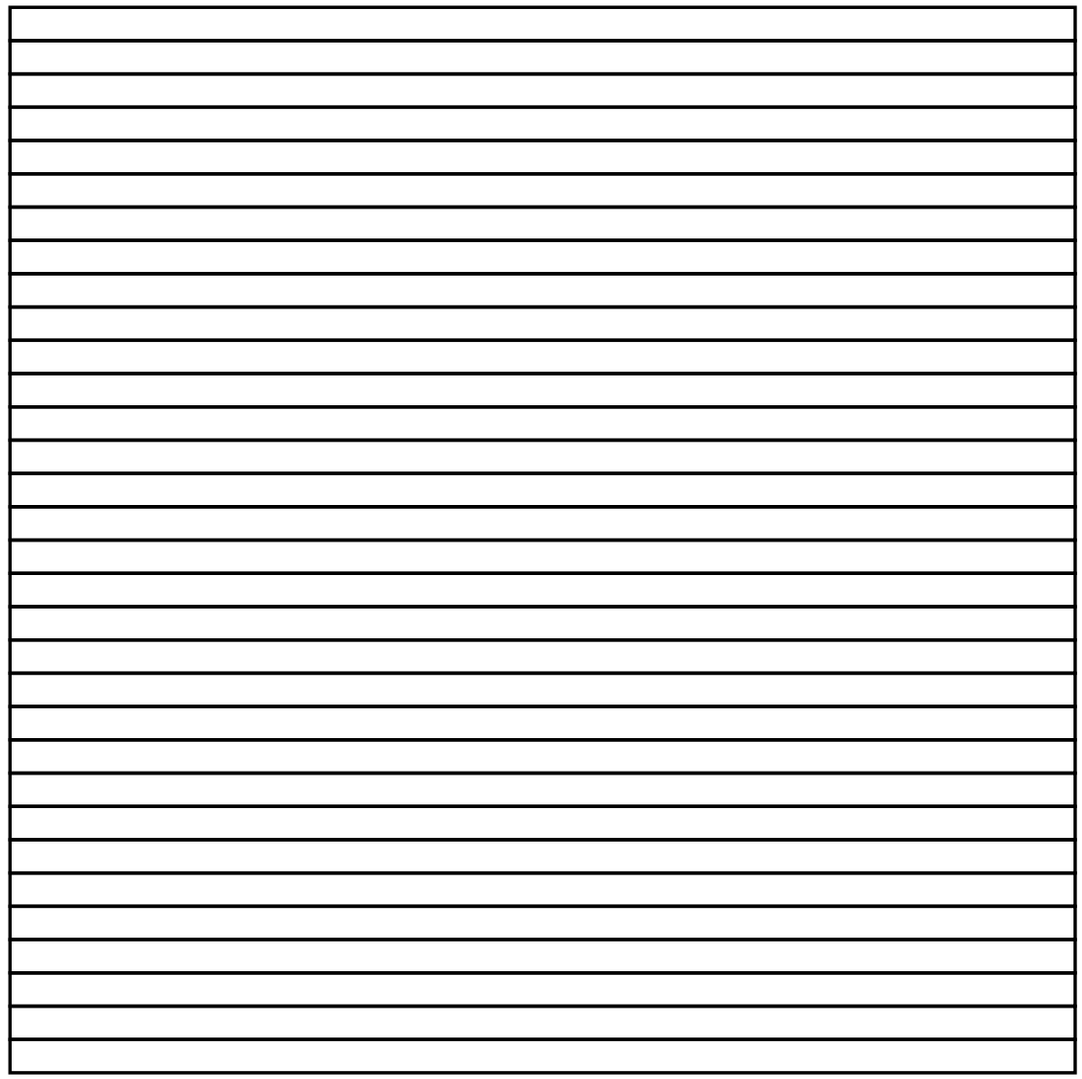


Actual Analysis

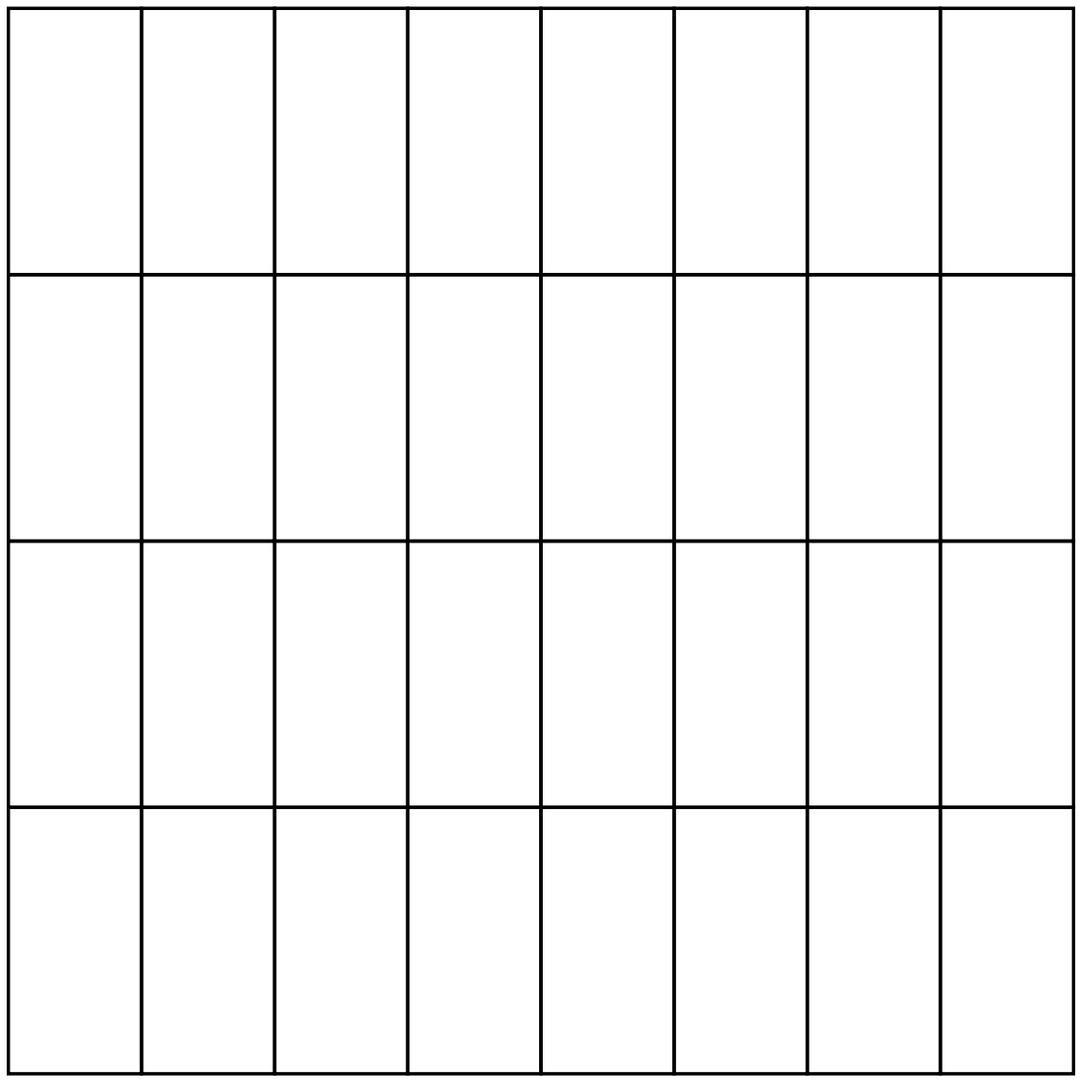


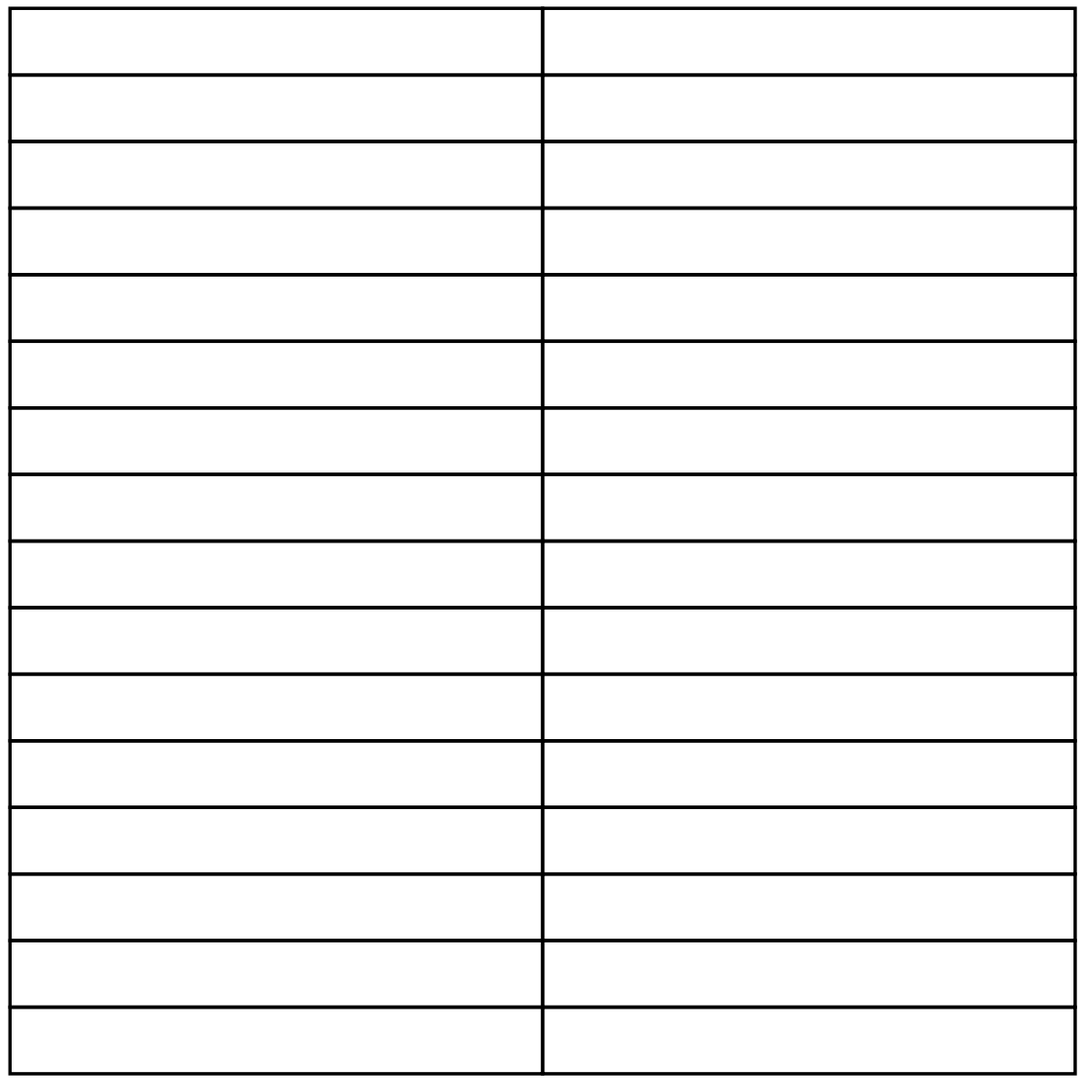
Dirac and Fourier Information Plane Tilings



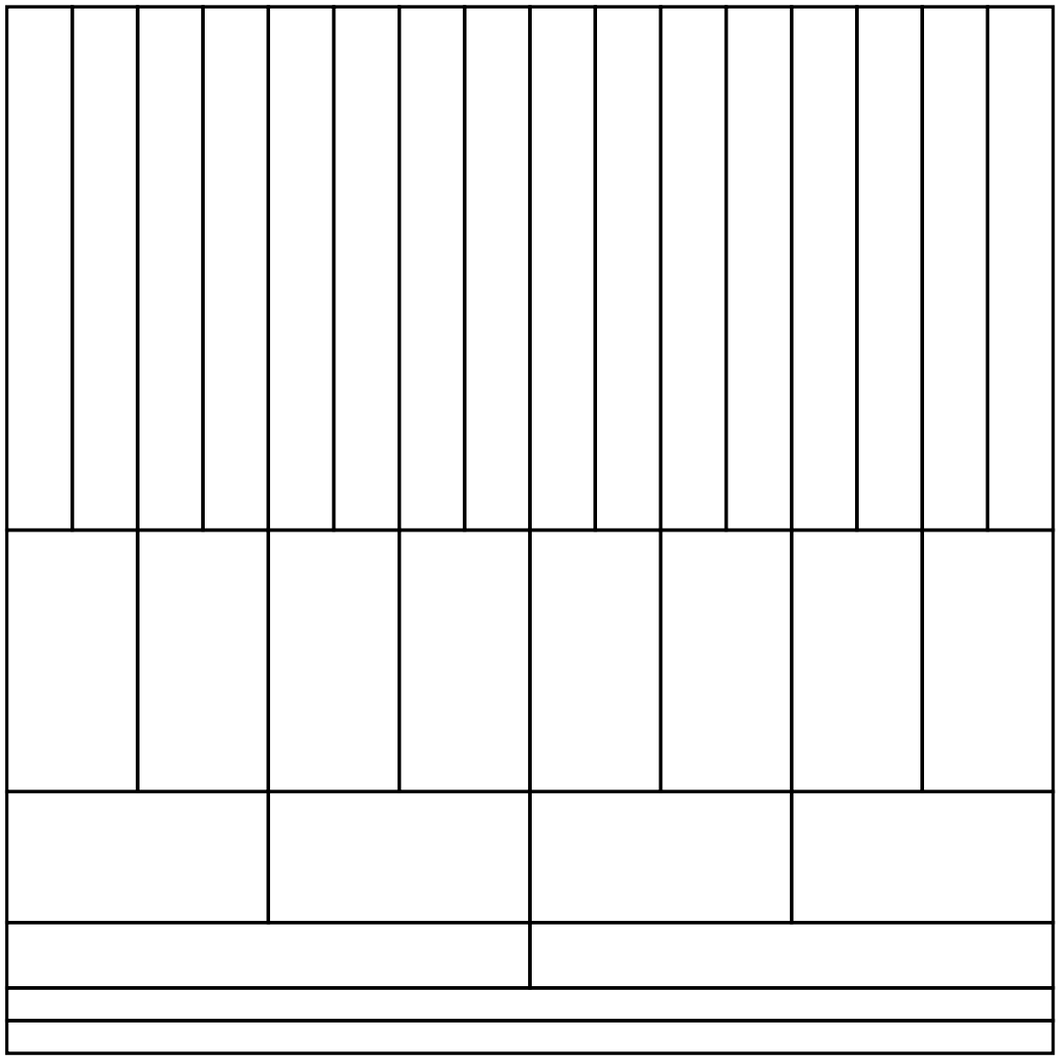


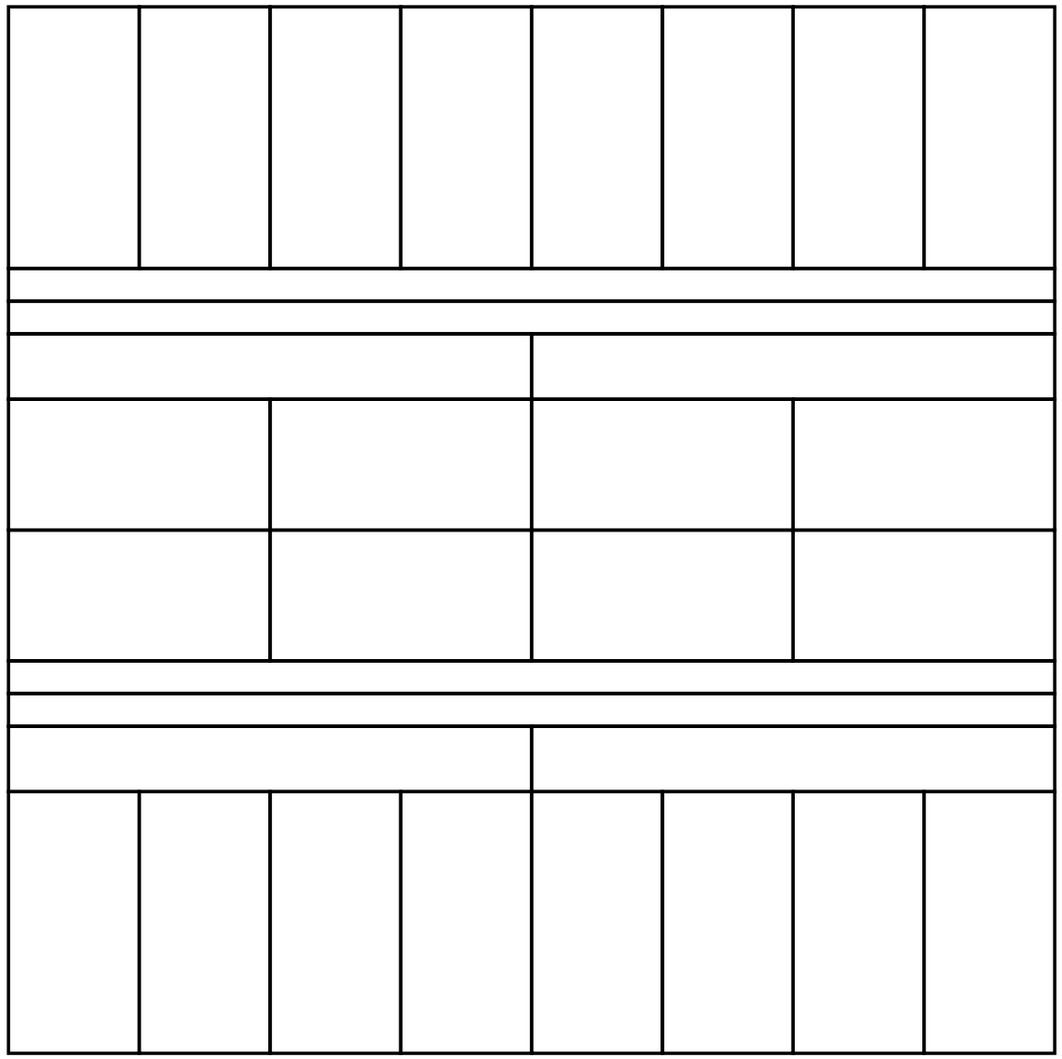
STFT Tilings of the Information Plane



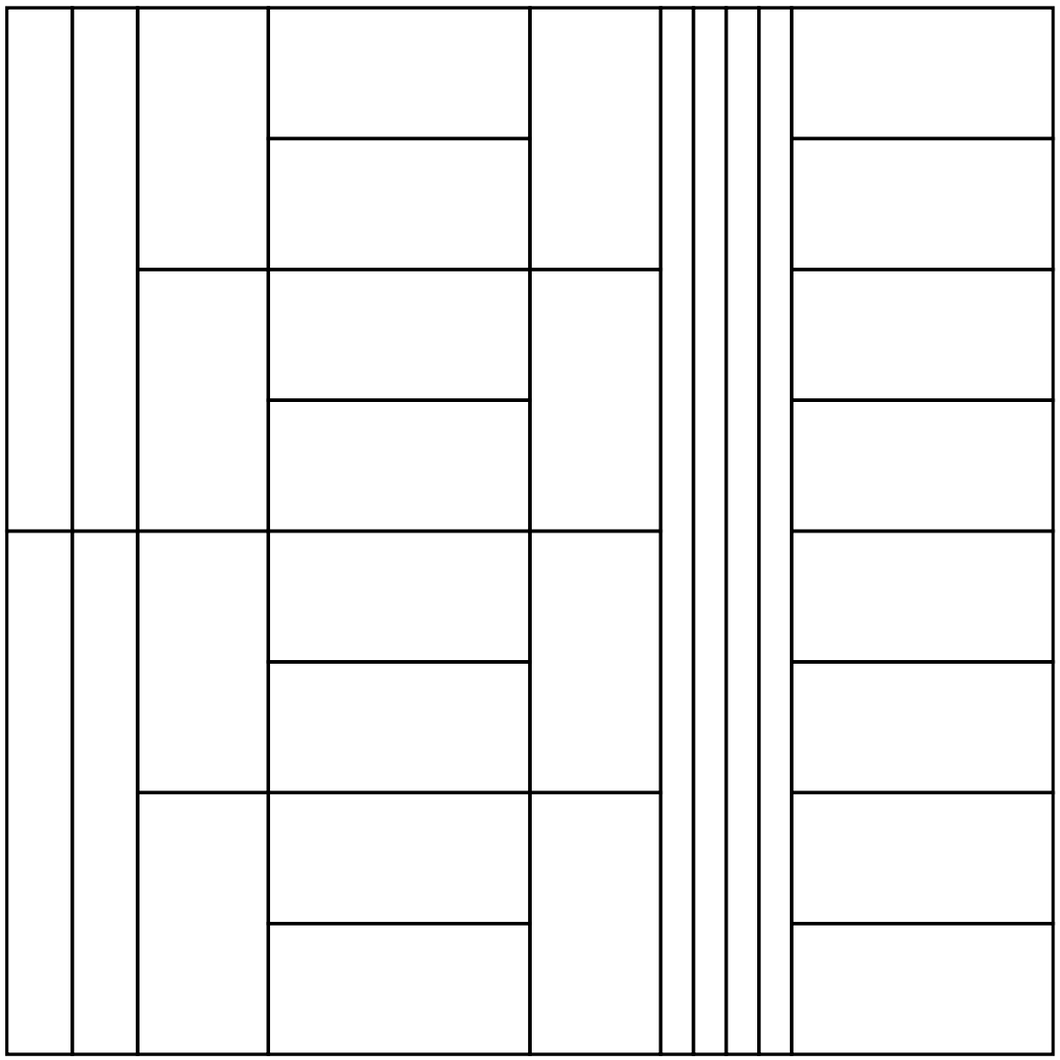


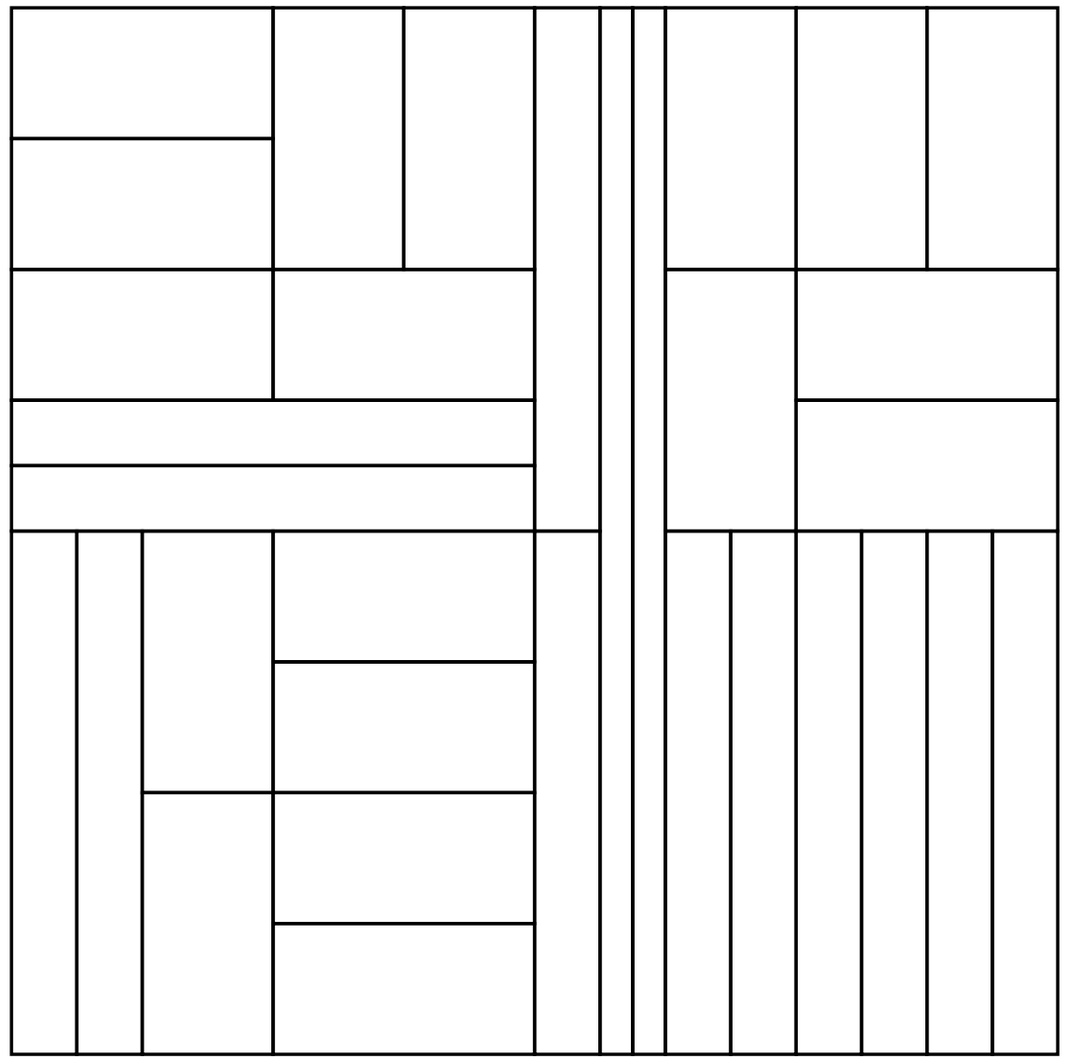
Wavelet and Wavelet Packet Tilings



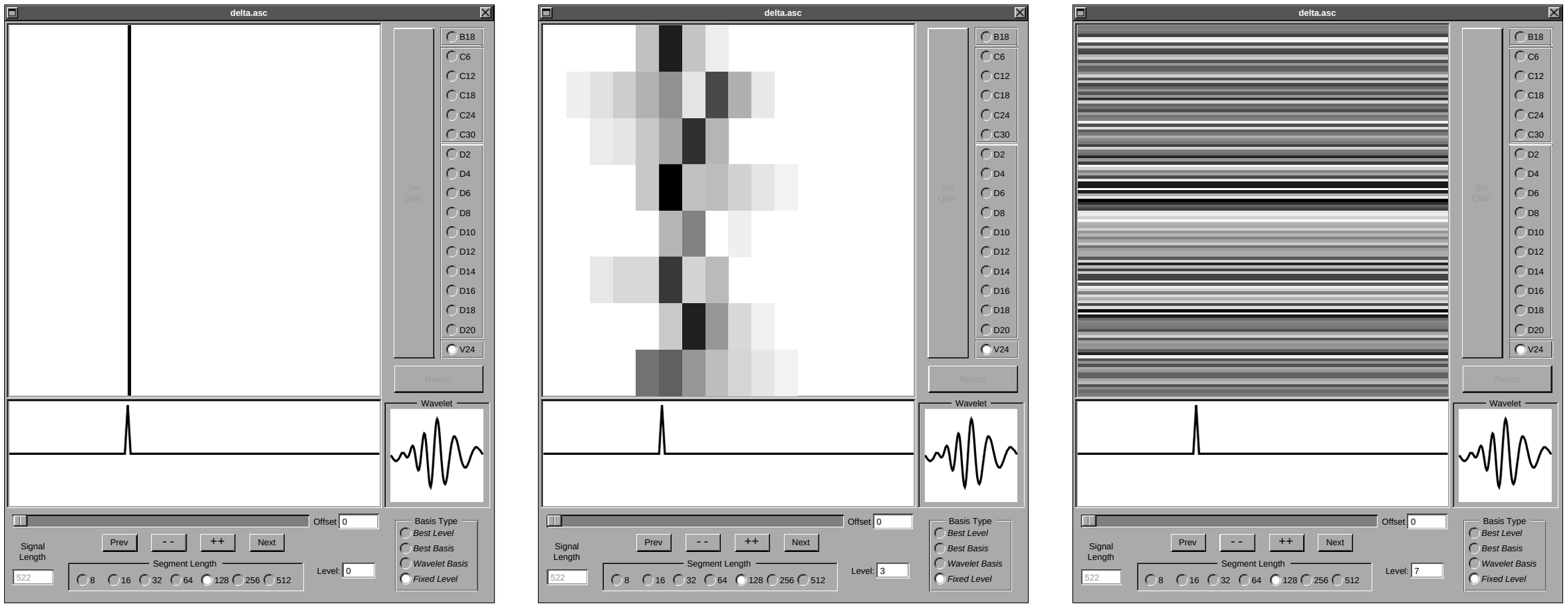


Adapted LOT and General Dyadic Tilings



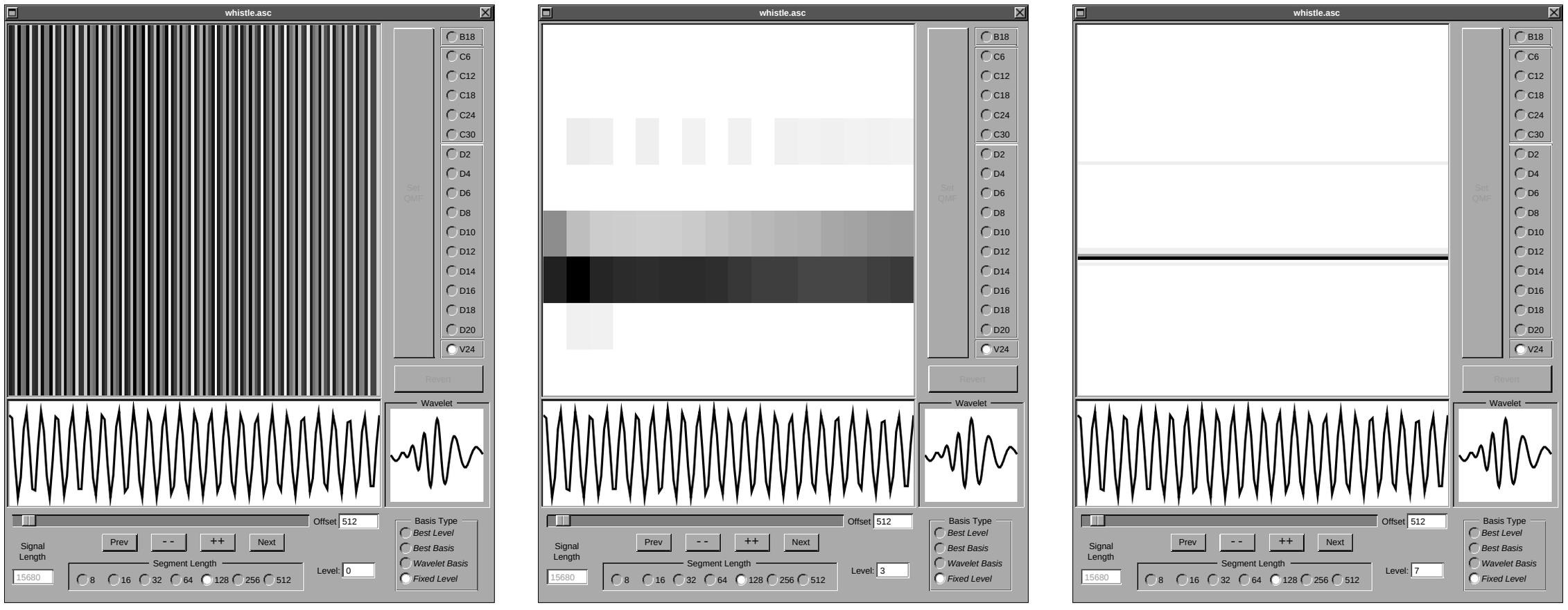


Impulse Analysis in Different Tilings



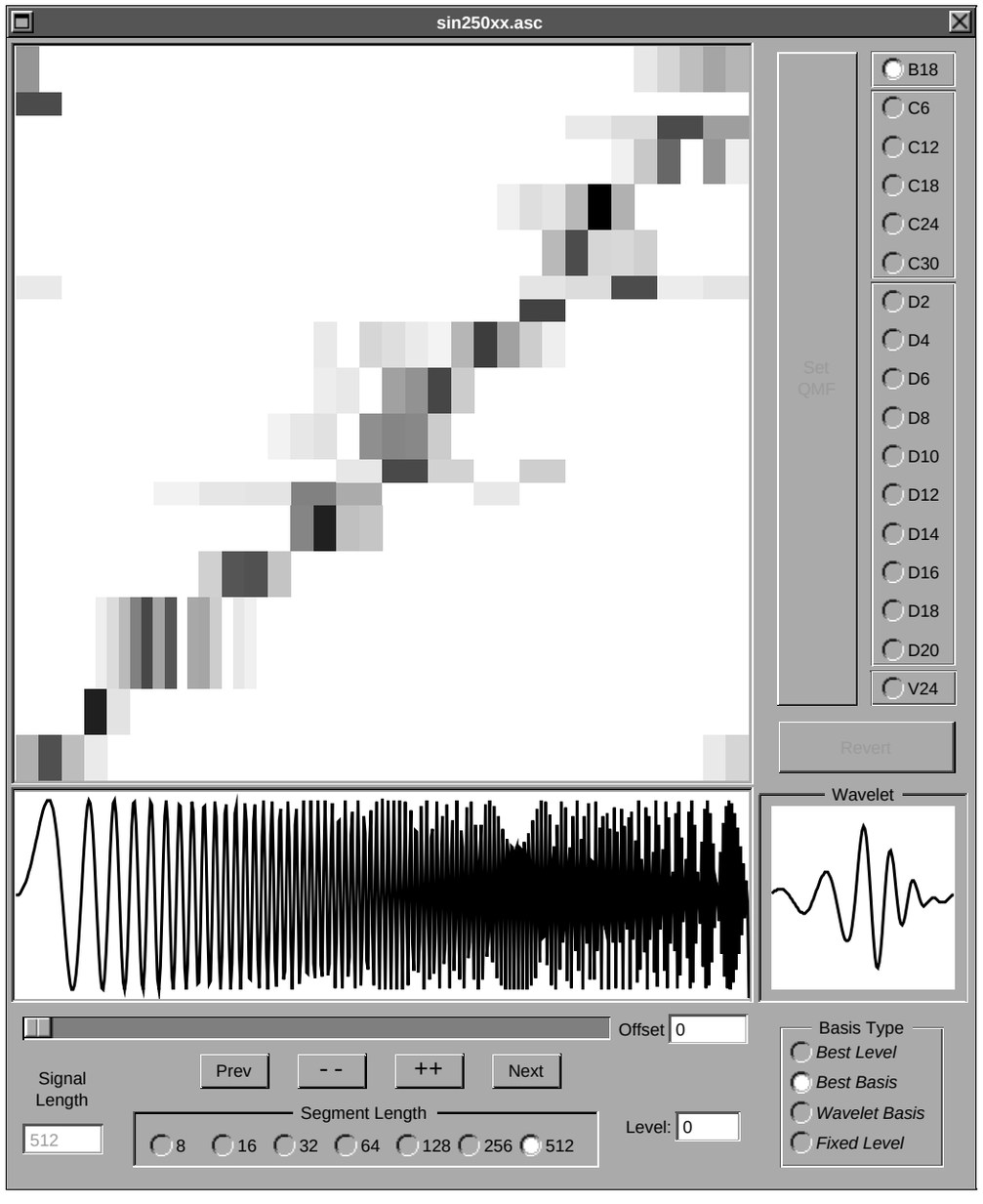
Impulse at increasing Fourier window sizes.

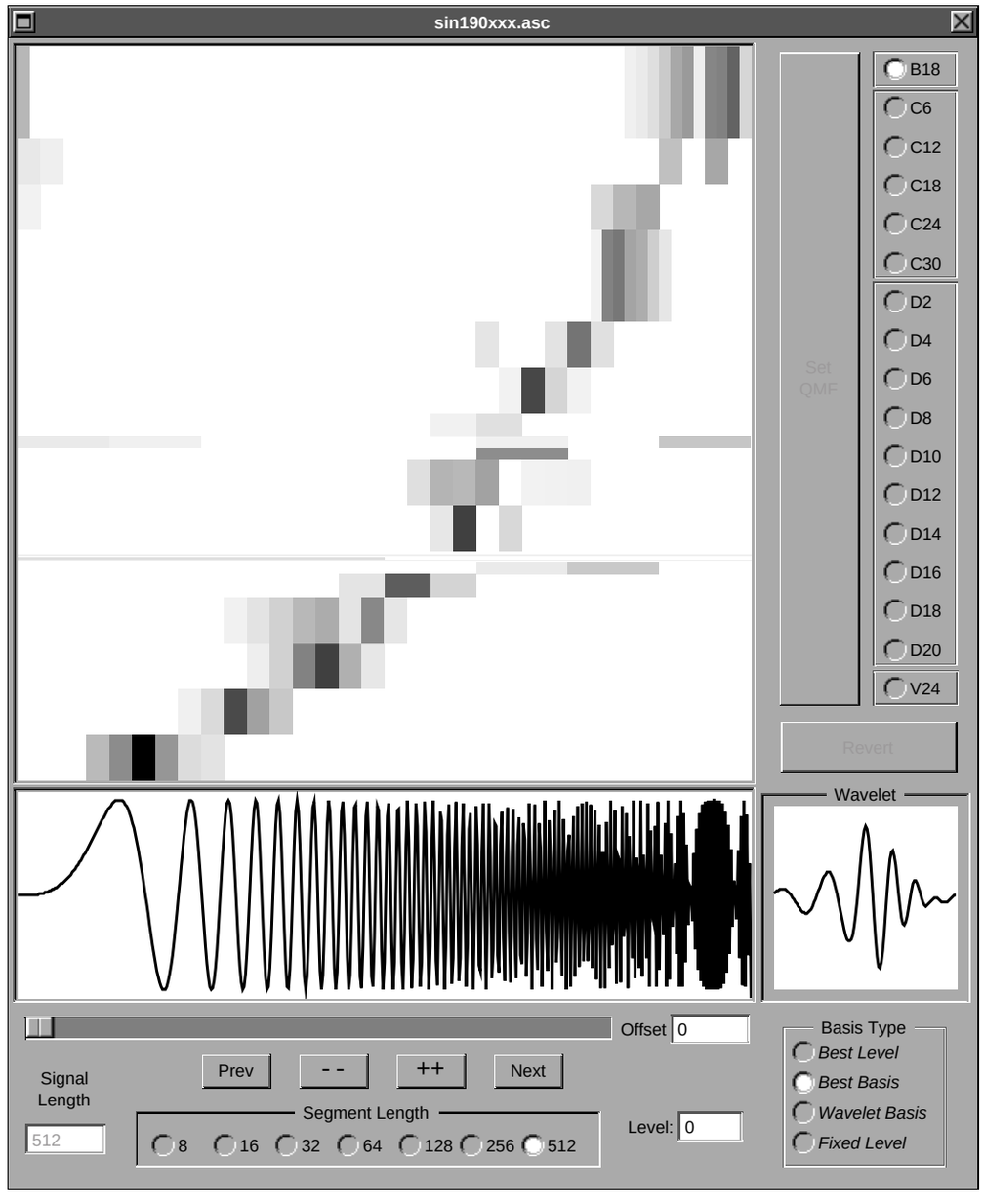
Tone Analysis in Different Tilings



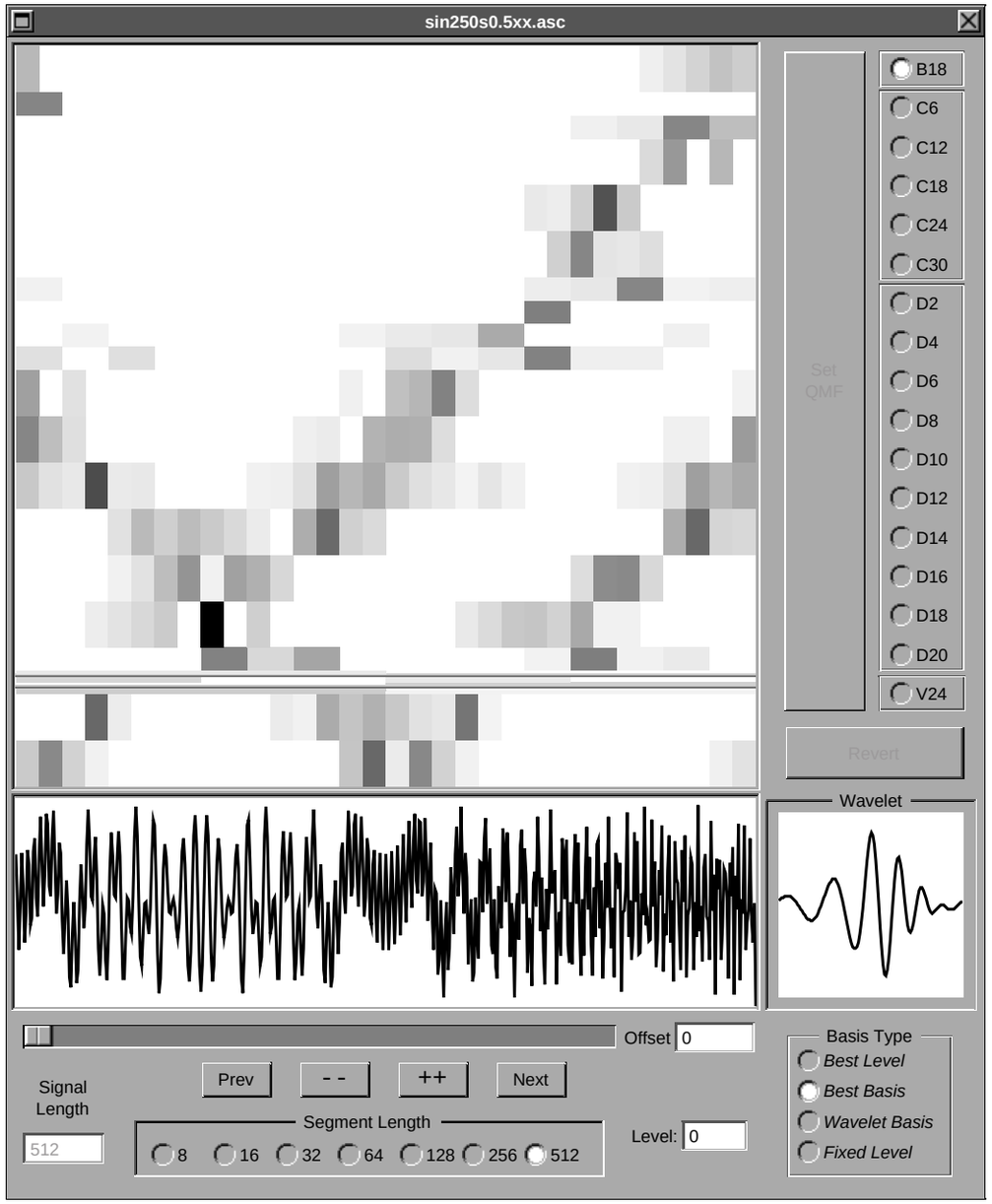
Whistle at increasing Fourier window sizes.

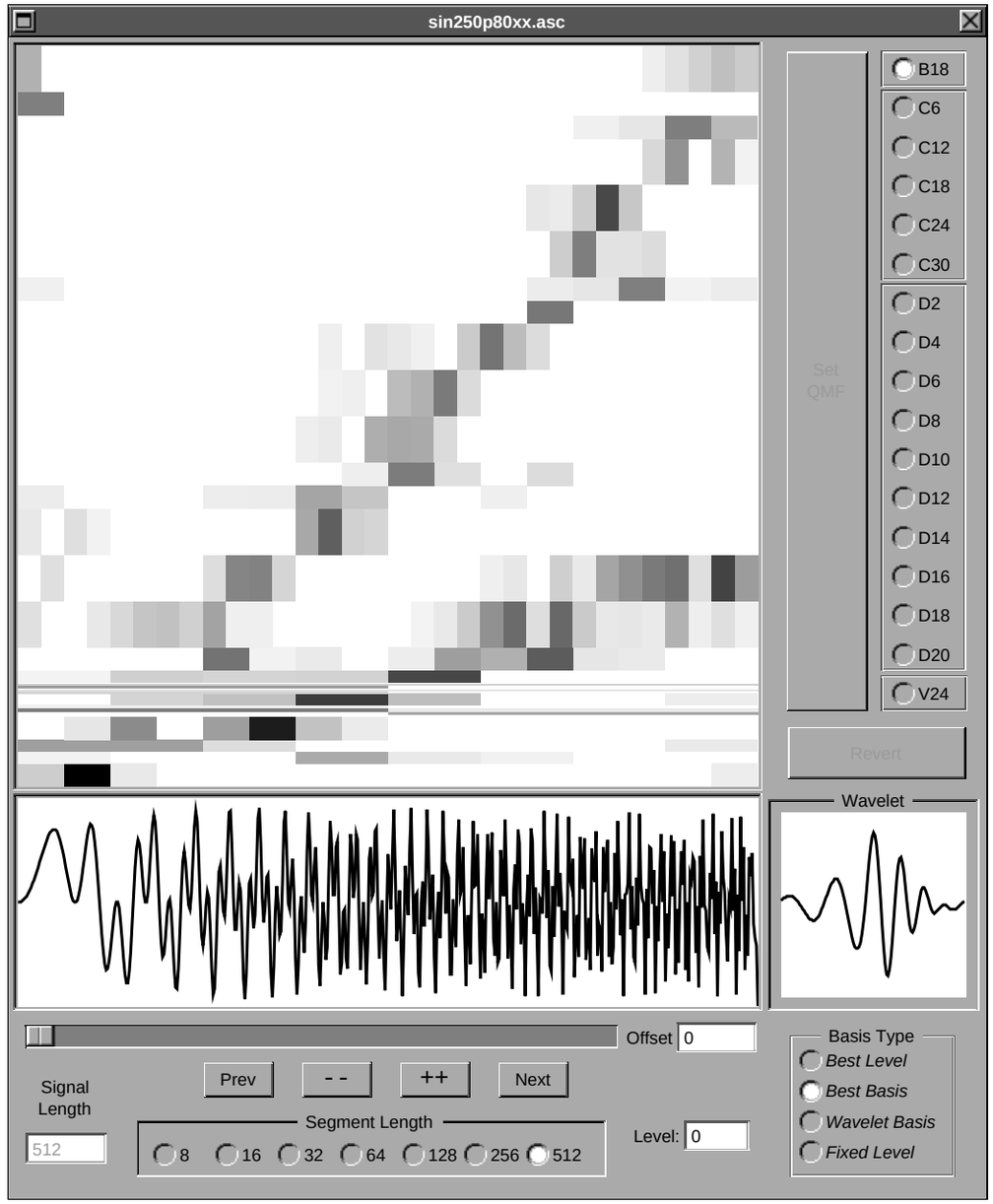
Chirp Analysis in Best Bases



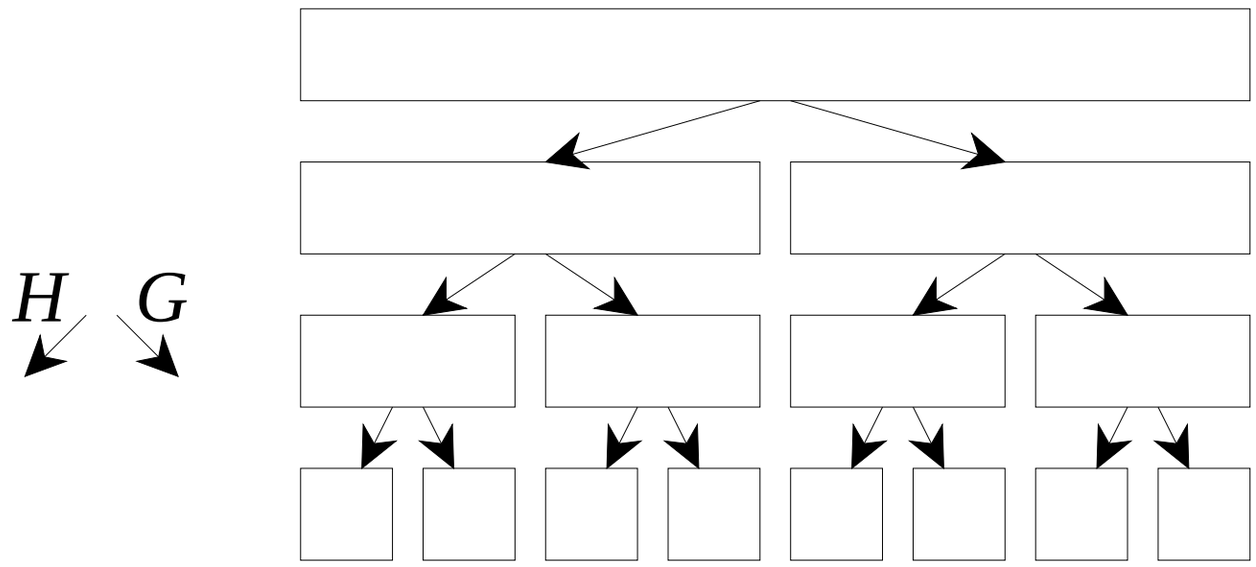


Superposed Chirp Analysis in Different Tilings





Recursive Splitting Algorithms



General Conditions on H, G:

- $HH^* = I$ and $GG^* = I$, so H^*H and G^*G are orthogonal projections;
- $HG^* = GH^* = 0$, so H and G project onto independent subspaces;
- $H^*H + G^*G = I$, so H and G together allow perfect reconstruction.

Example: Haar-Walsh splitting

Define

$$\begin{aligned} Hx(n) &= [x(2n) + x(2n+1)]/2; \\ Gx(n) &= x(2n+1) - x(2n). \end{aligned}$$

$$H^*x(n) = \begin{cases} x(\frac{n}{2}), & \text{if } n \text{ is even;} \\ x(\frac{n-1}{2}), & \text{if } n \text{ is odd;} \end{cases}$$

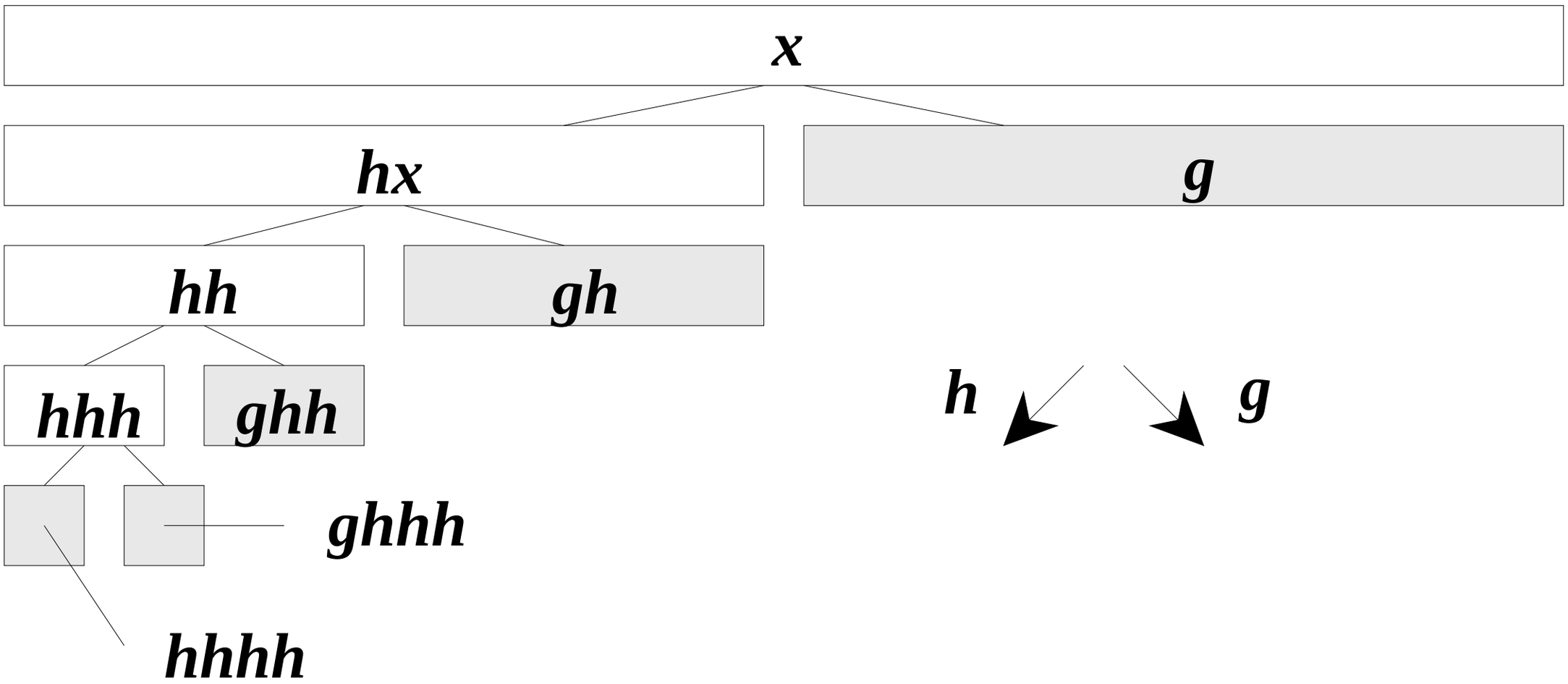
$$G^*x(n) = \begin{cases} -\frac{1}{2}x(\frac{n}{2}), & \text{if } n \text{ is even;} \\ \frac{1}{2}x(\frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

Thus

$$\begin{aligned} HH^*x(n) &= [H^*x(2n) + H^*x(2n+1)]/2 \\ &= [x(n) + x(n)]/2 \;=\; x(n); \\ GG^*x(n) &= G^*x(2n+1) - G^*x(2n) \\ &= \frac{1}{2}x(n) - [-\frac{1}{2}x(n)] \;=\; x(n). \end{aligned}$$

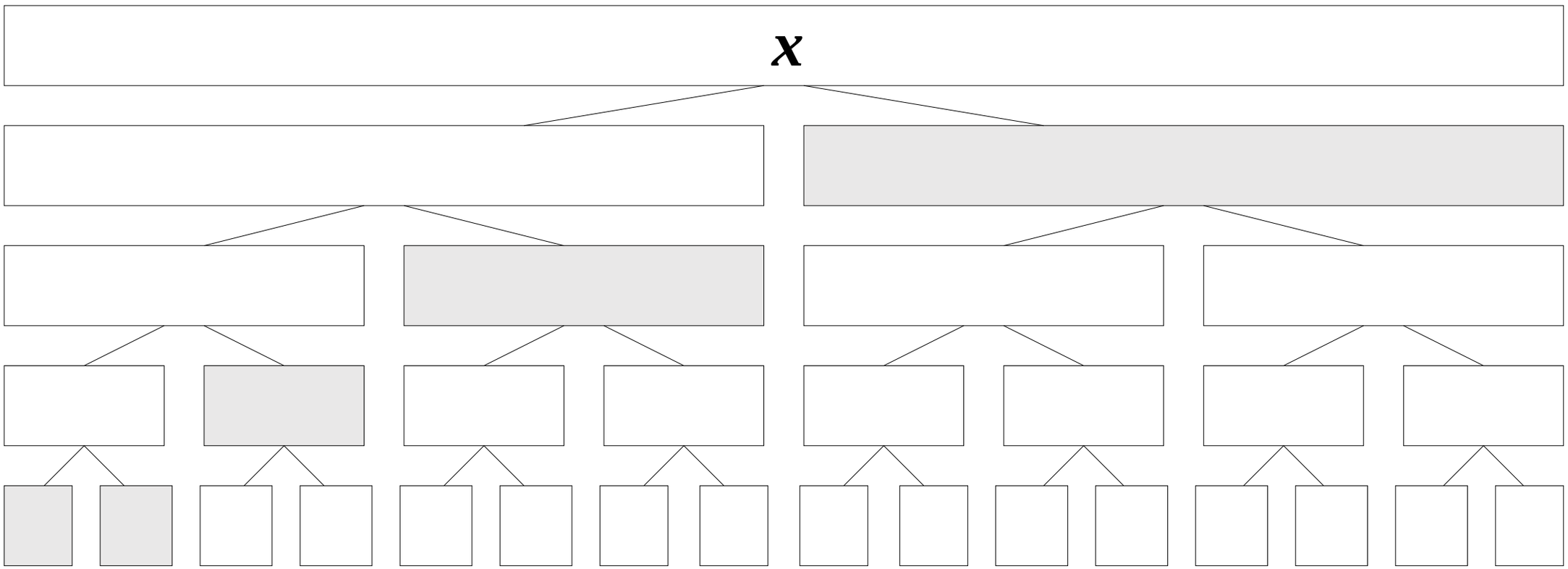
$$\begin{aligned} x(n) &= \begin{cases} Hx(\frac{n}{2}) - \frac{1}{2}Gx(\frac{n}{2})), & n \text{ even;} \\ Hx(\frac{n-1}{2}) + \frac{1}{2}Gx(\frac{n-1}{2}), & n \text{ odd,} \end{cases} \\ &= H^*Hx(n) + G^*Gx(n). \end{aligned}$$

Discrete Wavelet Transform (DWT) 1



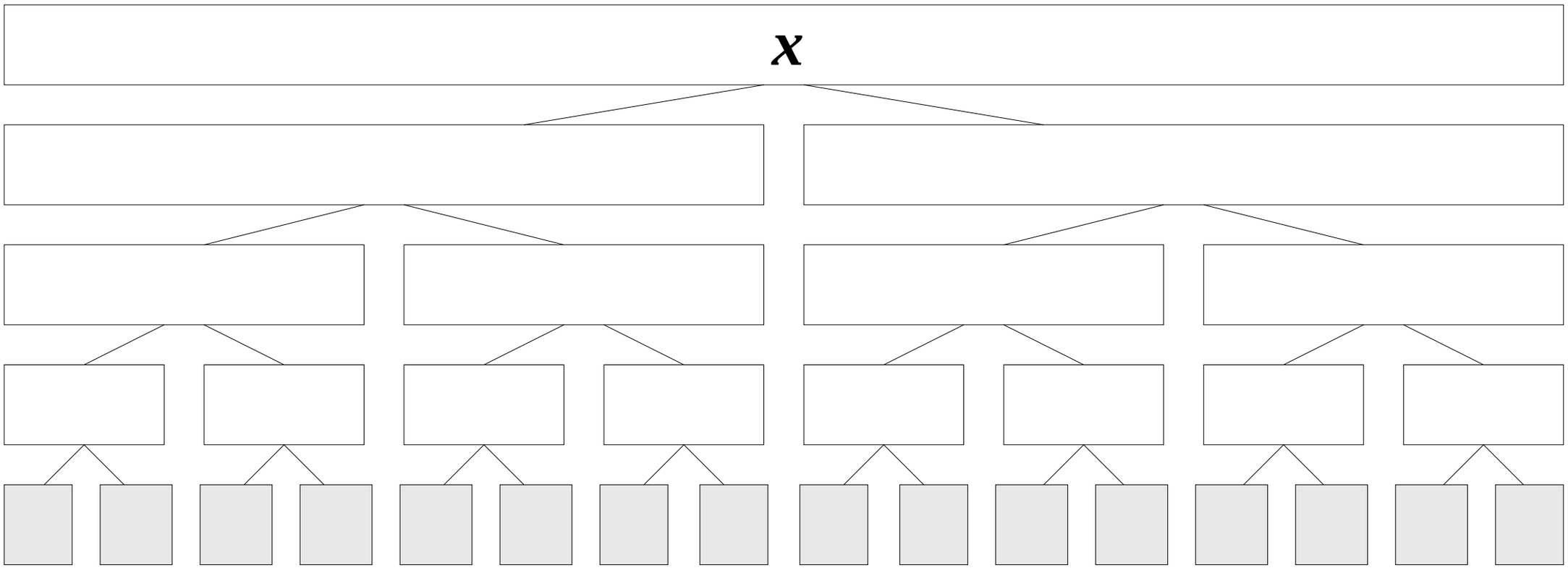
Mallat's original multiresolution DWT...

Discrete Wavelet Transform 2



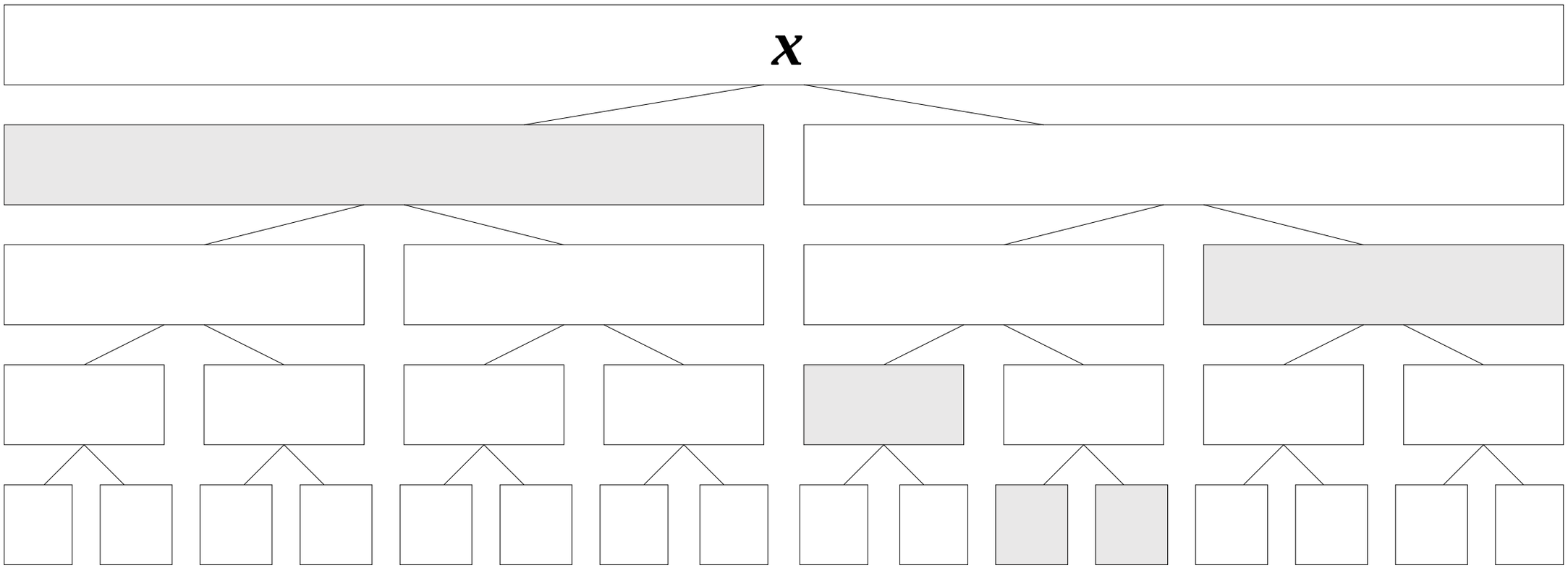
. . . embedded in a discrete wavelet packet decomposition.

Discrete Wavelet Packet Transform 1



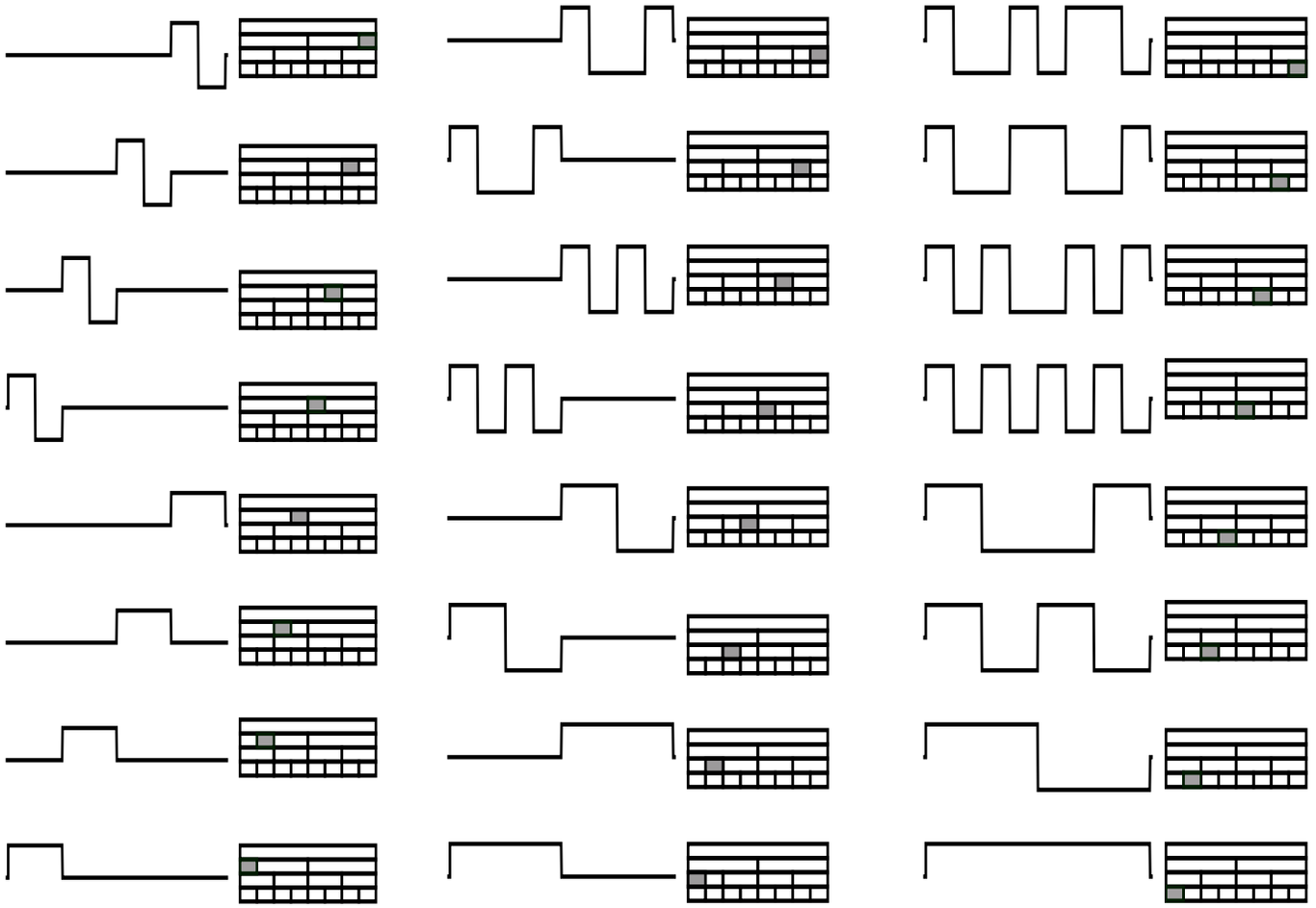
Complete subband, or Walsh-type transform.

Discrete Wavelet Packet Transform 2

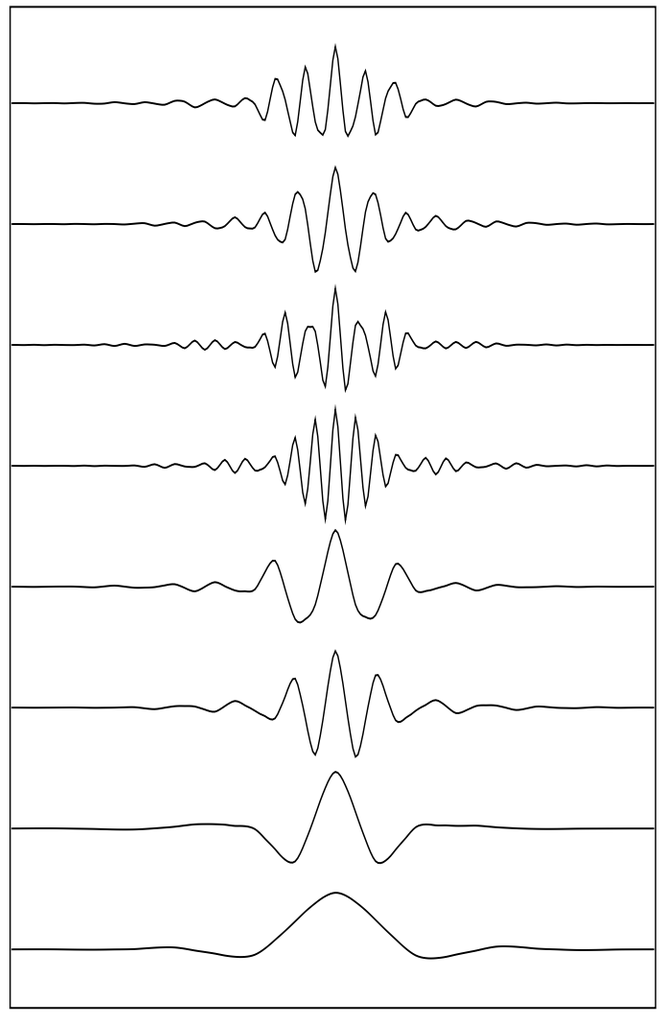


Yet another basis in the wavelet packet library.

Underlying Functions

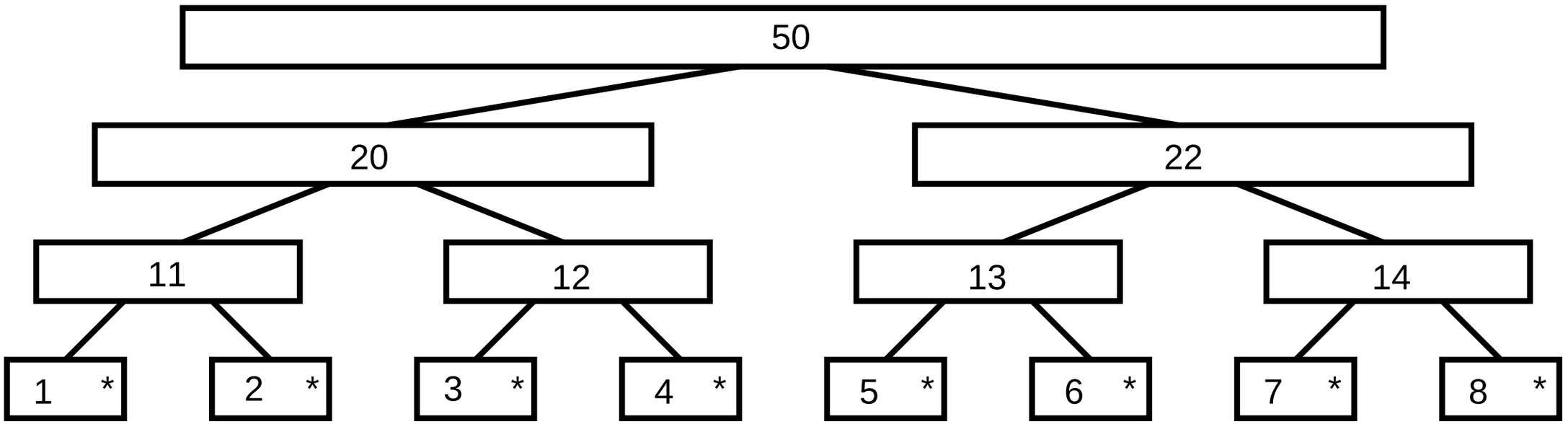


All Haar-Walsh



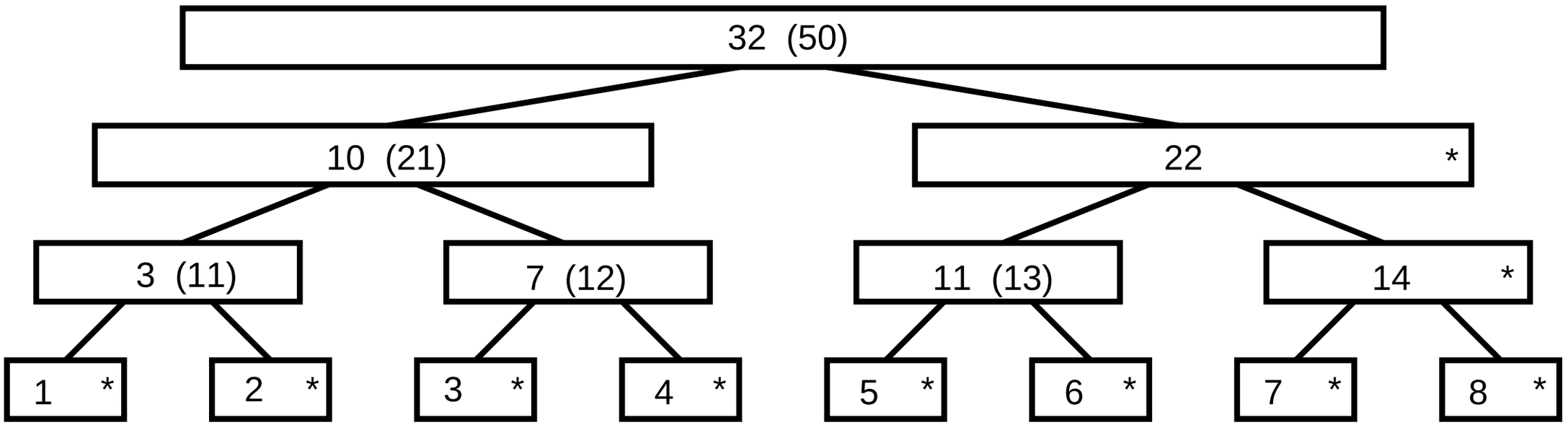
Coiflet 30

Best Basis Search 1



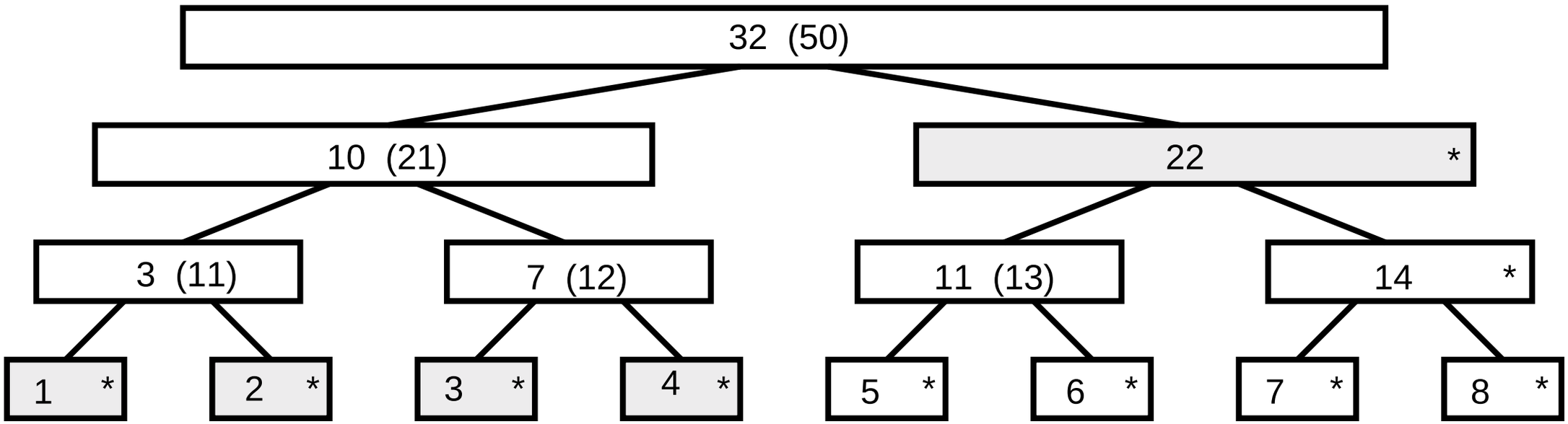
First stage: compute costs, mark leaves.

Best Basis Search 2



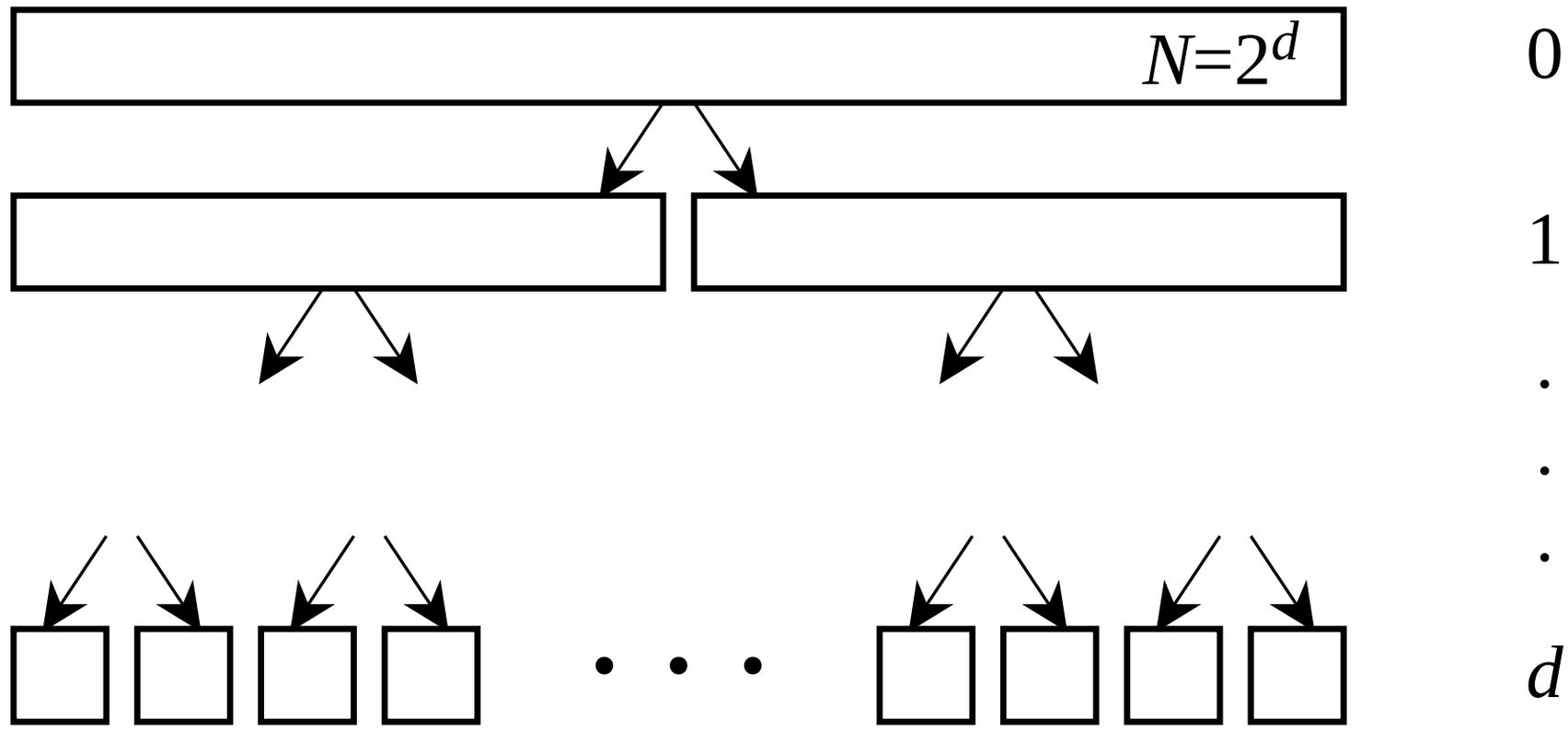
Middle: mark nodes better than descendents.

Best Basis Search 3



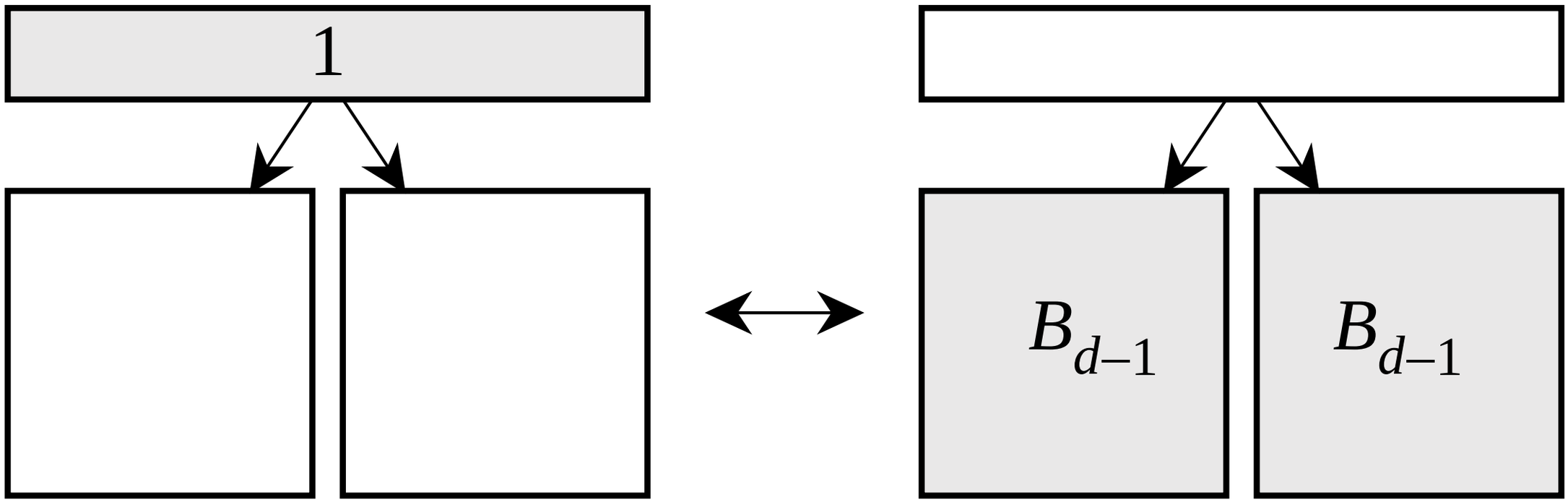
Final stage: keep topmost marked nodes.

How Many Graph Bases 1



Depth d — $d+1$ levels — B_d bases.

How Many Graph Bases 2

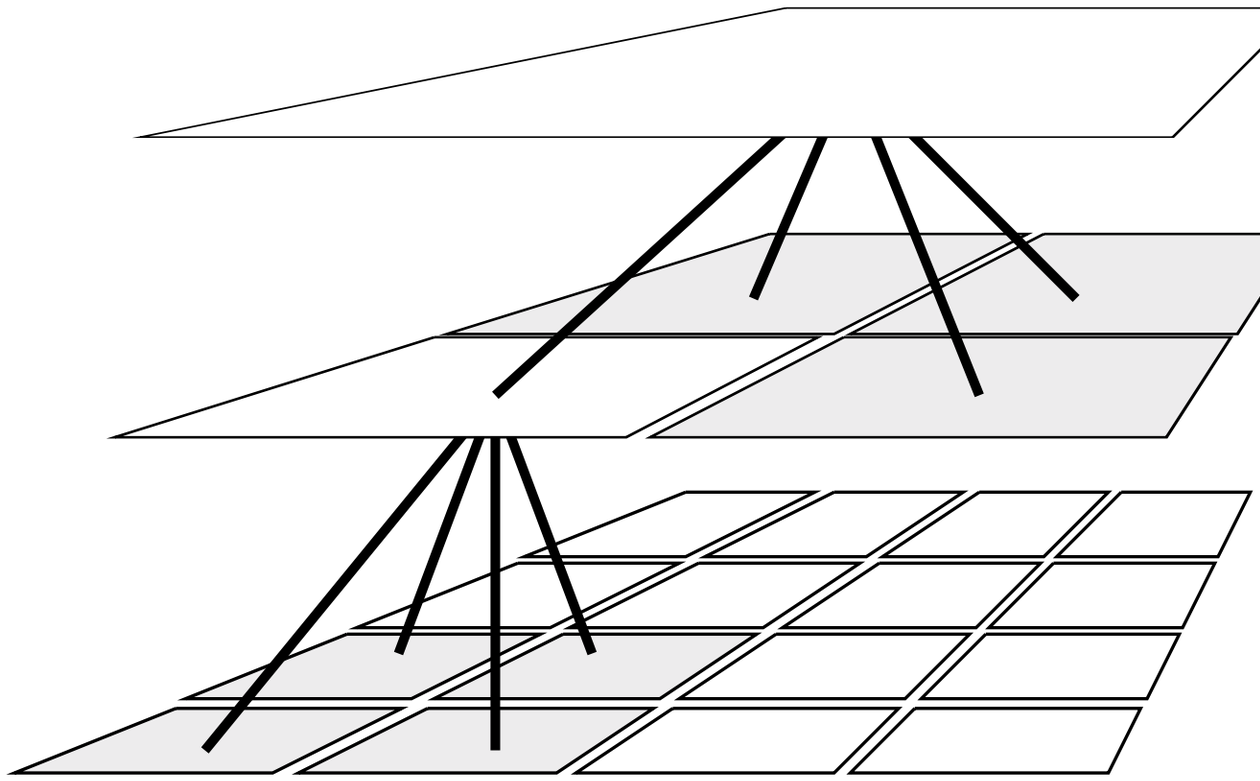


Recursion $B_d = 1 + B_{d-1}^2 > B_{d-1}^2$, with $B_1 = 2$, implies

$$B_d > 2^{2^d} = 2^N, \qquad d > 1.$$

Two-Dimensional Splitting

Apply operators H, G separately for multidimensions:



Dimension 2: quadtree to depth 2.

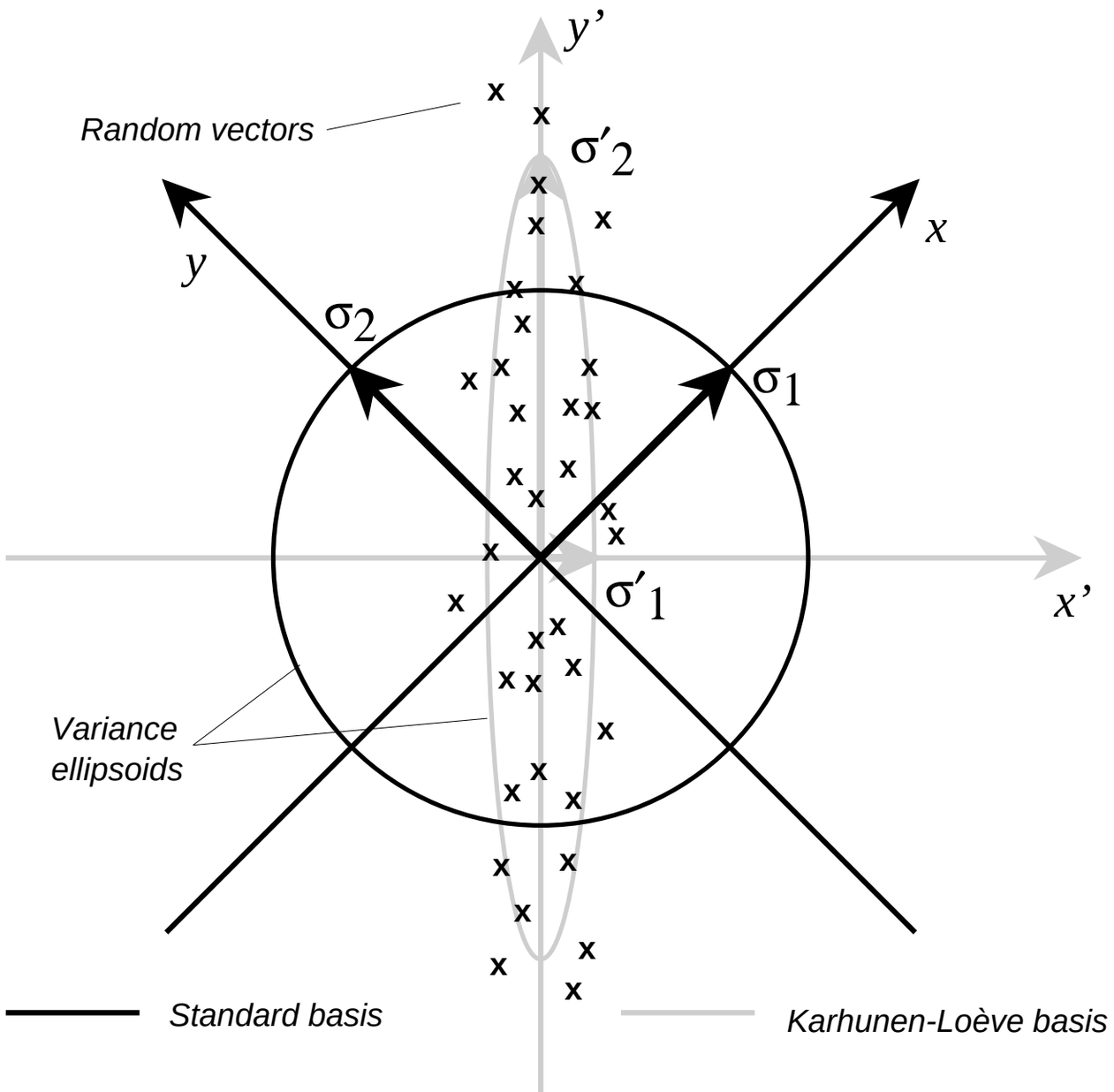
In D dimensions, each step will produce 2^D descendents.

Example: Face Images



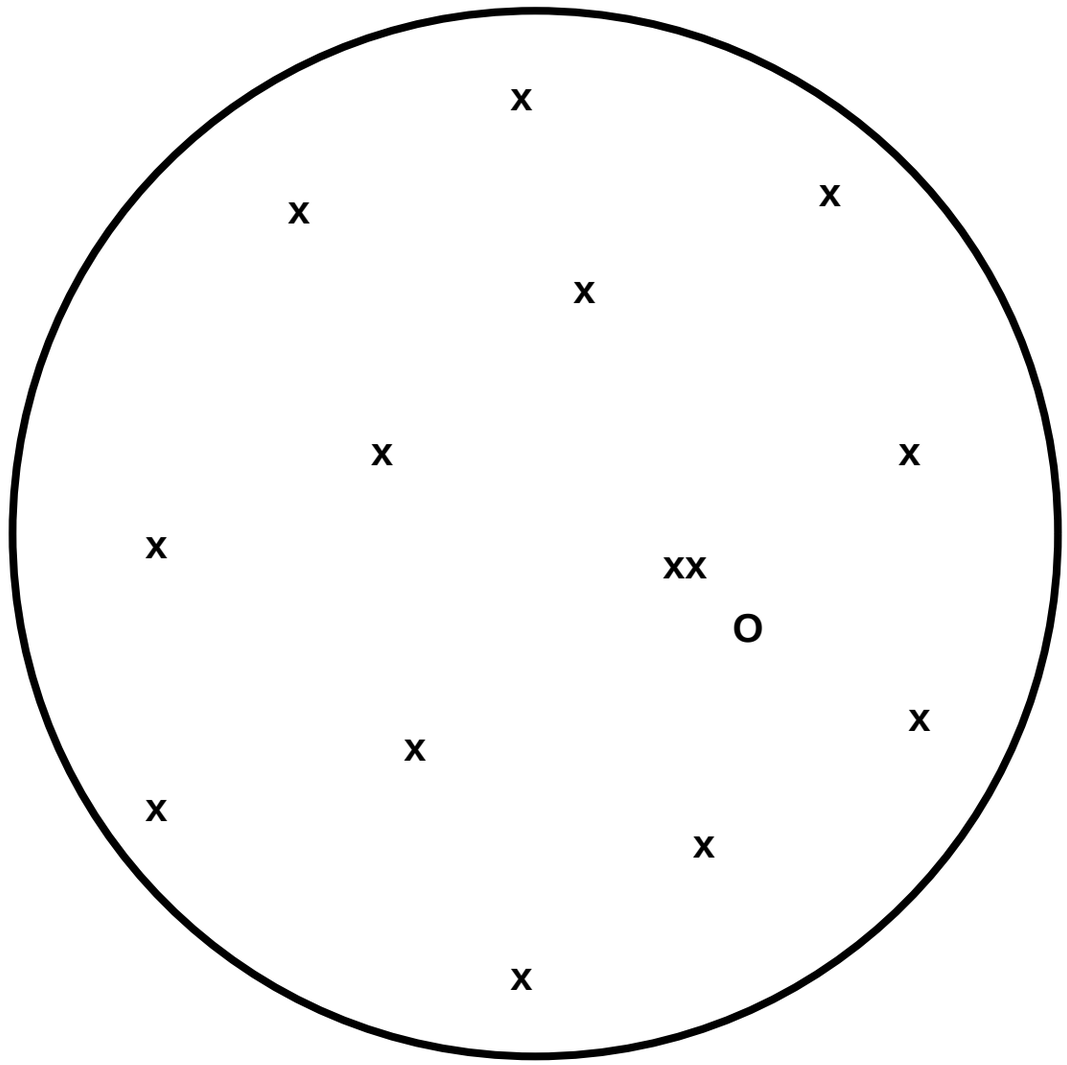
Face minus average face yields caricature.

Application 1: Data Compression



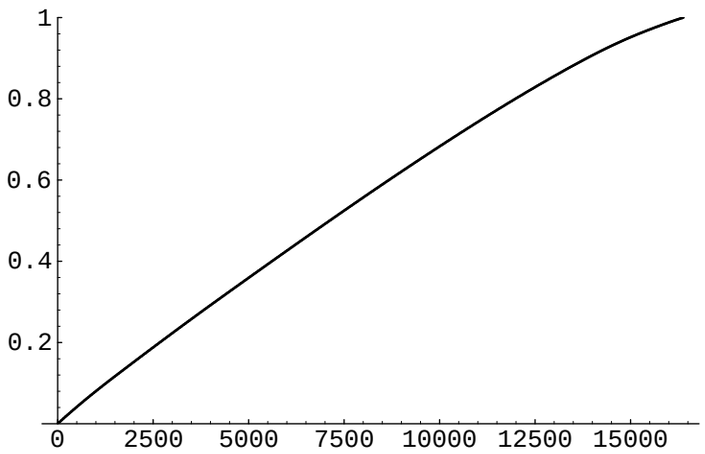
KL: Choose coordinates to concentrate variance.

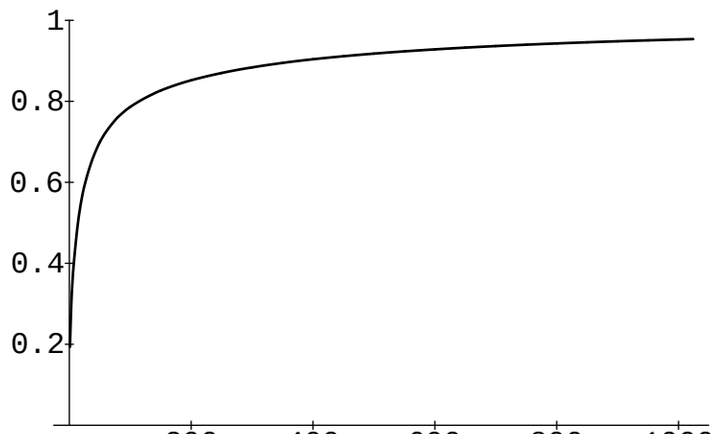
Fast KL

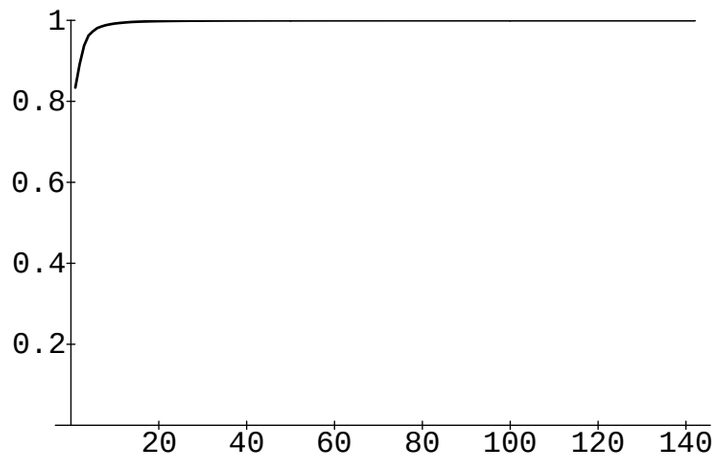


Choose only among low-complexity transforms.

Fast Transforms Work Pretty Well

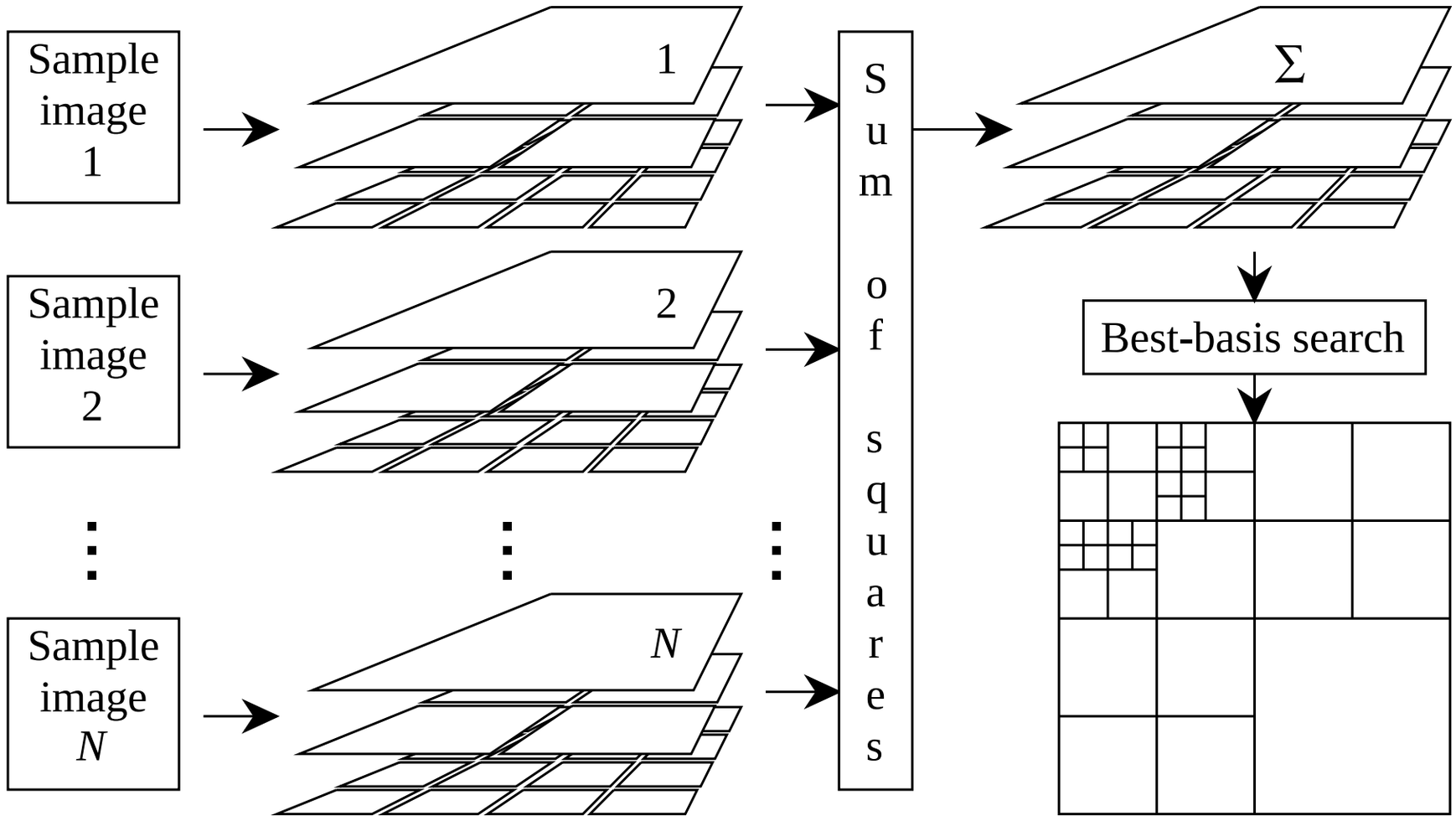






Variance in original, fast KL, and KL coordinates.

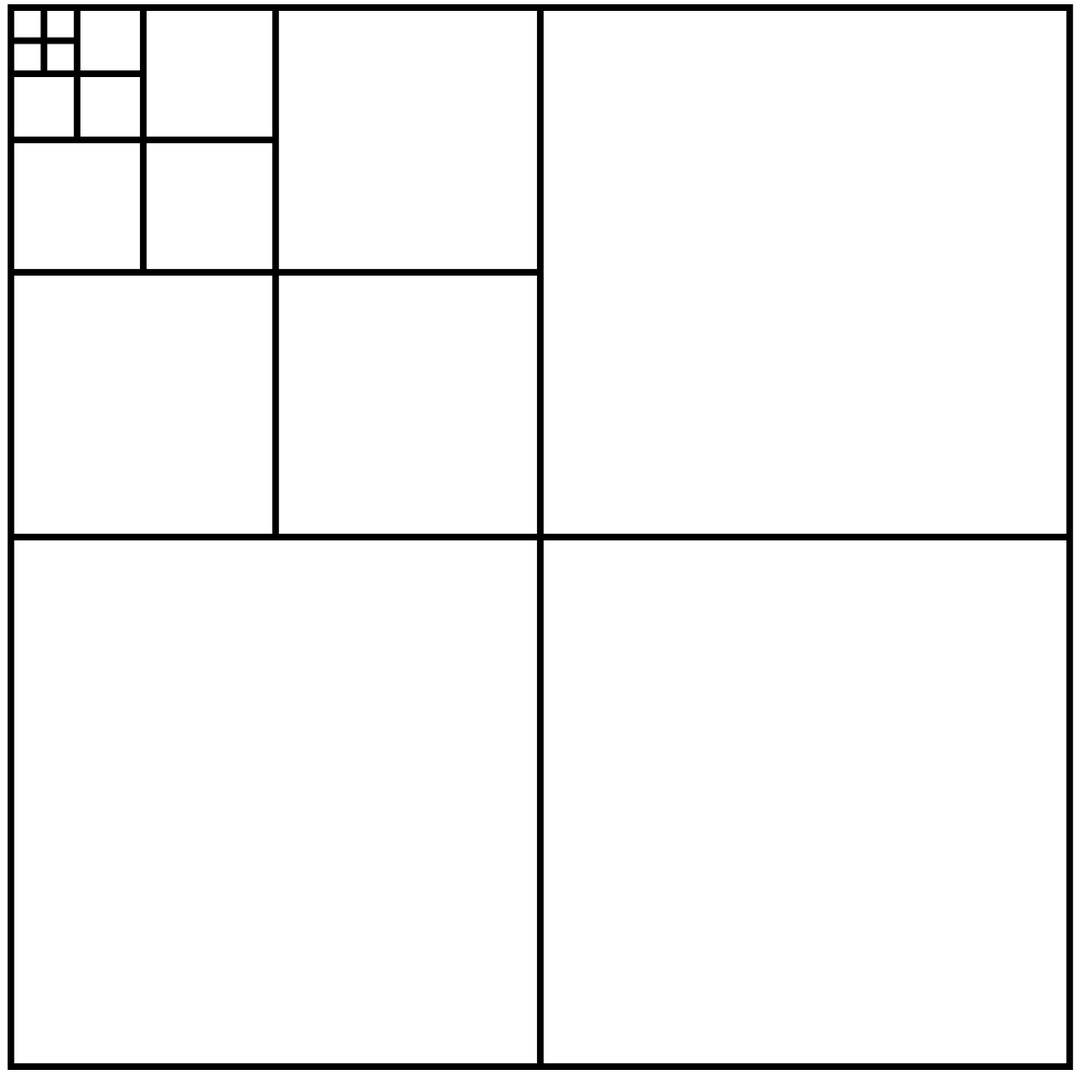
Joint Best-Basis



Joint best basis (JBB) training algorithm:

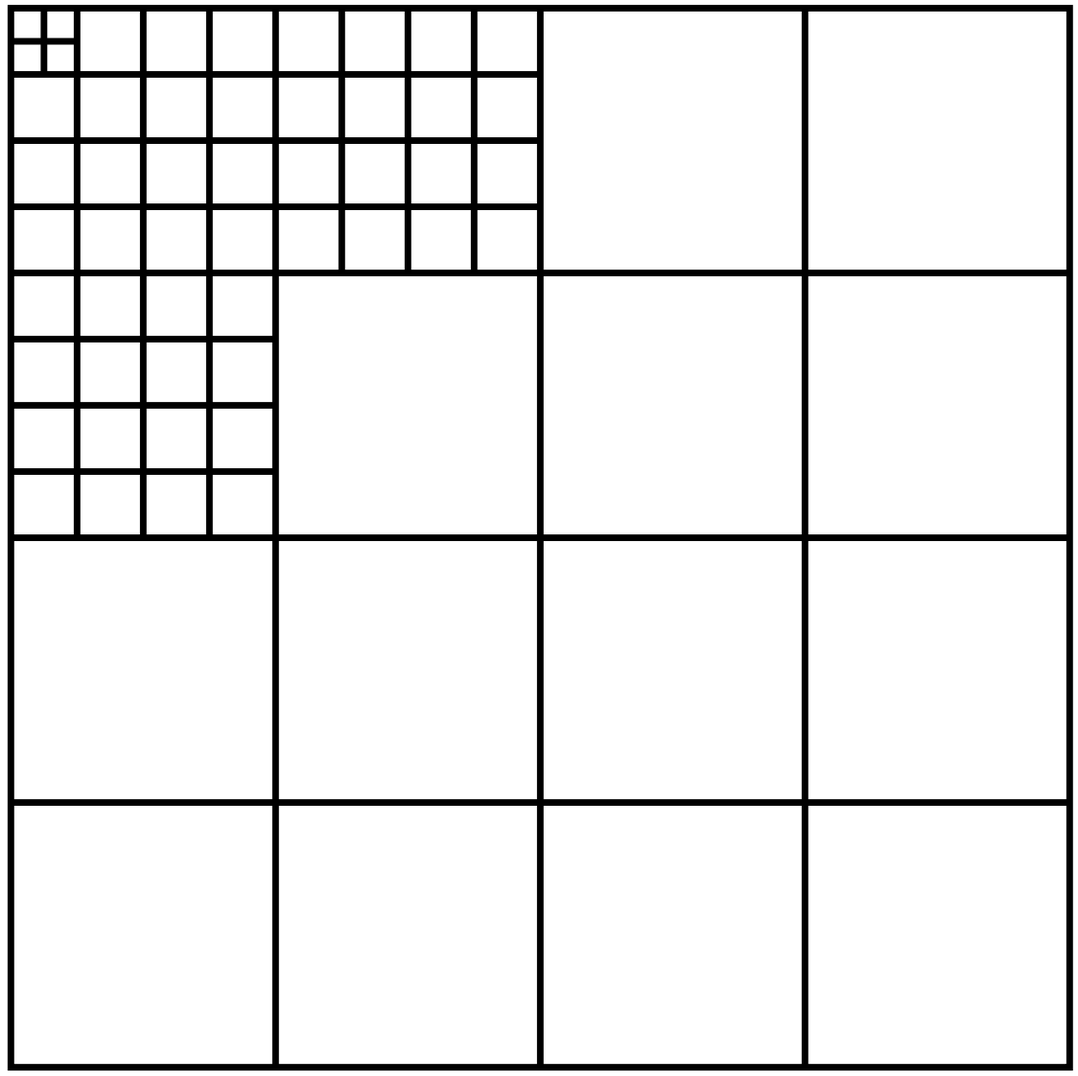
1. expand all training images in all bases
2. determine coordinate variances in all bases
3. search for JBB using variance concentration cost function
4. keep top few JBB coordinates plus basis description

Good Bases for Images 1



5-level wavelet basis, used in JPEG-2000.

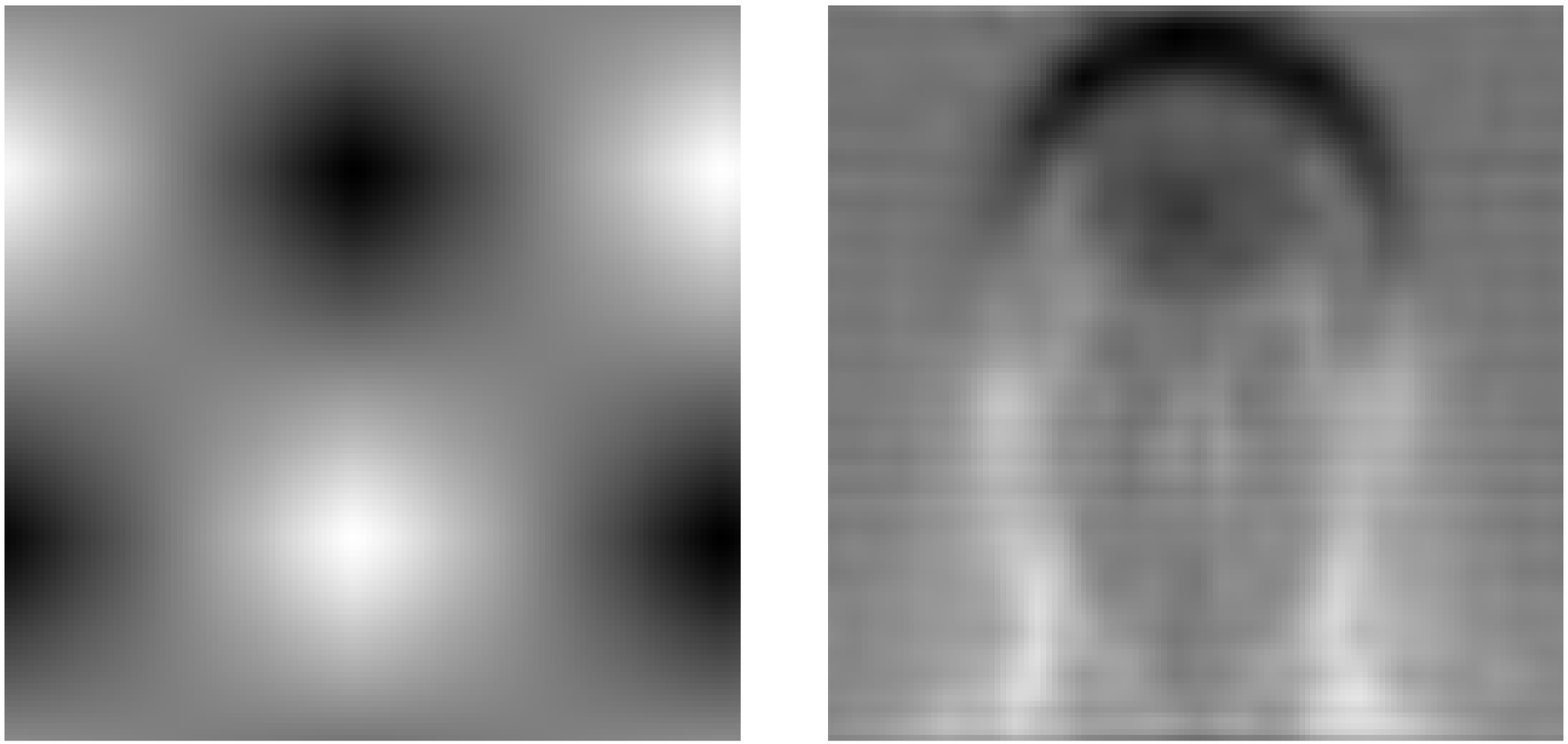
Good Bases for Images 2



5-level wavelet packet basis, used in WSQ.

Application 2: Classification

PROBLEM: Given a training set divided into two classes A and B, find a wavelet packet basis that maximizes a discriminant function.



Left: Wavelet from JBB. Right: KL eigenface.

Wavelet Features for Classification

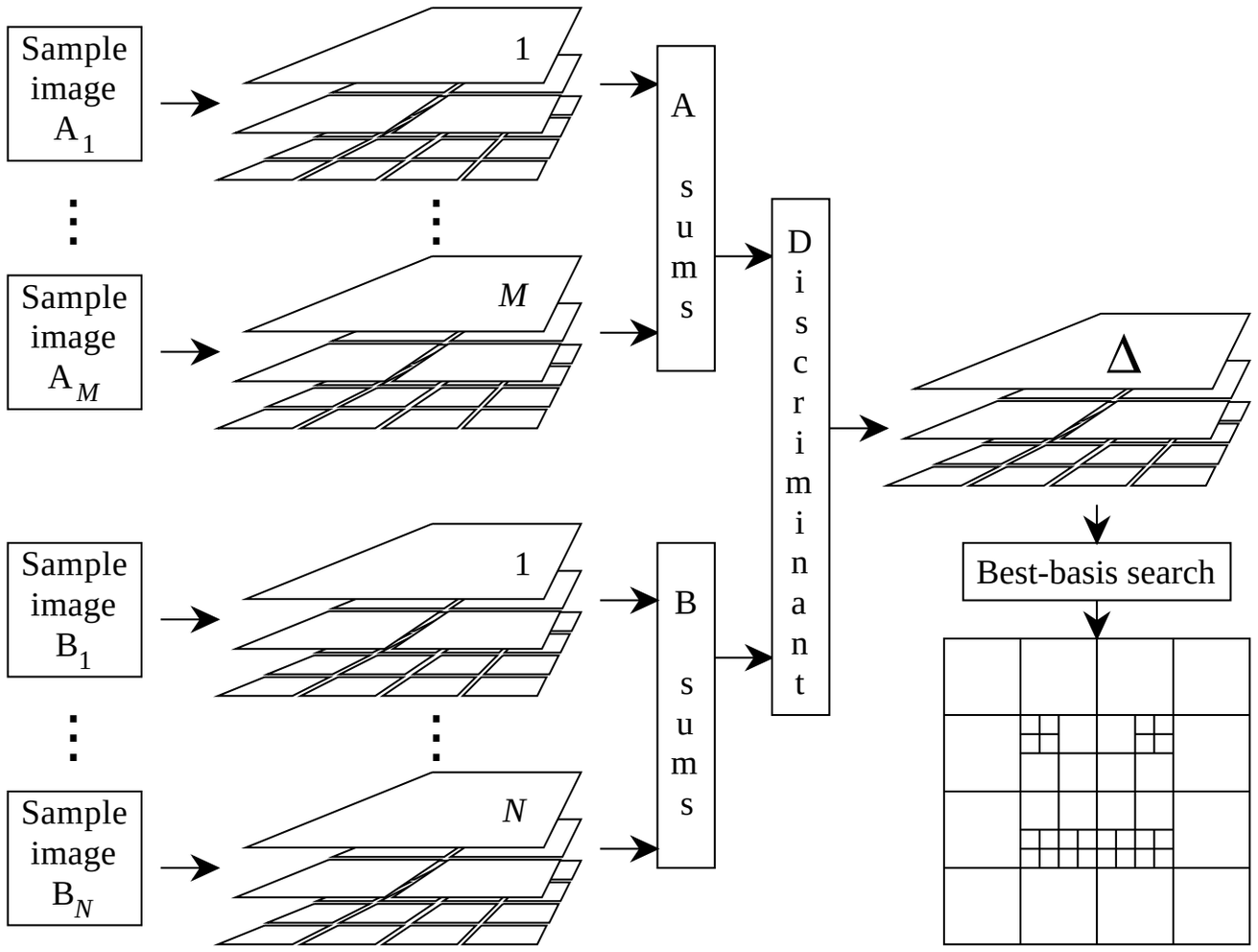
Advantages of wavelet features:

- nice basis functions
- fast pre-processing transforms

Difficulties with wavelet features:

- no shift invariance
- classifying features are non-intuitive

Local Discriminant Basis



Local discriminant basis (LDB) training:

1. expand all of both classes in all bases
2. determine coordinate discrimination power in all bases
3. search for LDB using discrimination power concentration as the cost function
4. keep top few LDB coordinates plus basis description