Wavelets in Image Compression

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THEORY AND APPLICATIONS OF WAVELETS
A Workshop Honoring **Ingrid Daubechies**
Recipient of the 2011 Benjamin Franklin Medal in Electrical Engineering
Villanova University, April 28th, 2011

Great Men from the Eighteenth Century I



Benjamin Franklin, 1706–1790.

Great Men from the Eighteenth Century II



Jean-Baptiste Joseph Fourier, 1768–1830.

Joseph Fourier's Construction

Theorem 1 *Any function $f = f(t)$ may be written as a sum of sines and cosines, multiplied by numbers $\{a_n, b_n\}$ specific to f:*

$$\begin{aligned} f(t) \;=\; & a_0 + a_1 \cos(t) + a_2 \cos(2t) + a_3 \cos(3t) + \cdots \\ & + b_1 \sin(t) + b_2 \sin(2t) + b_3 \sin(2t) + \cdots \end{aligned}$$

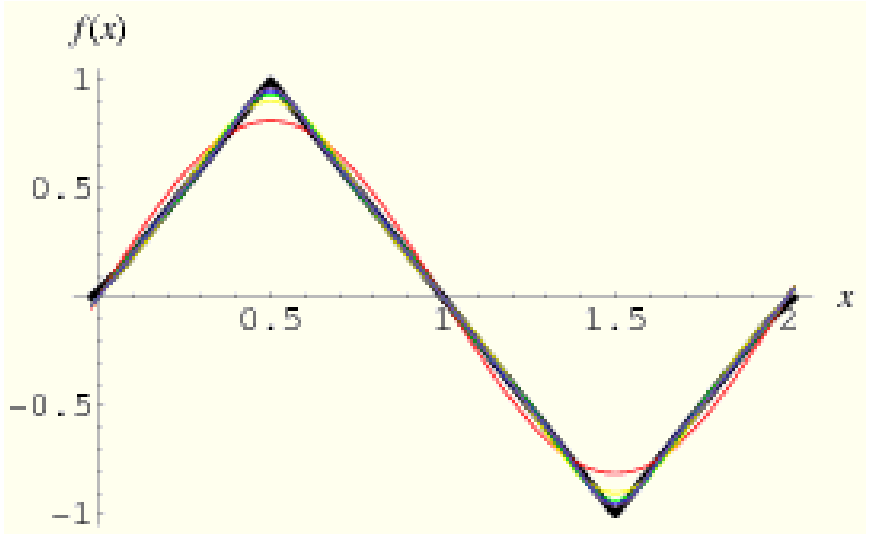
Key ideas:

- The building blocks are simple: sines and cosines.
- Data is simple: two numbers a_n, b_n for each frequency n.
- The system is complete and efficient, an orthonormal basis.

Application to Image Compression

- Images are functions.
- Functions are made of simple building blocks.
- Our senses are imperfect, so approximations suffice.
- Less accuracy requires less storage space.

Example of Fourier's Construction (Good)



Adding up just sines with $b_n \sim 1/n^2$ to get a sawtooth.

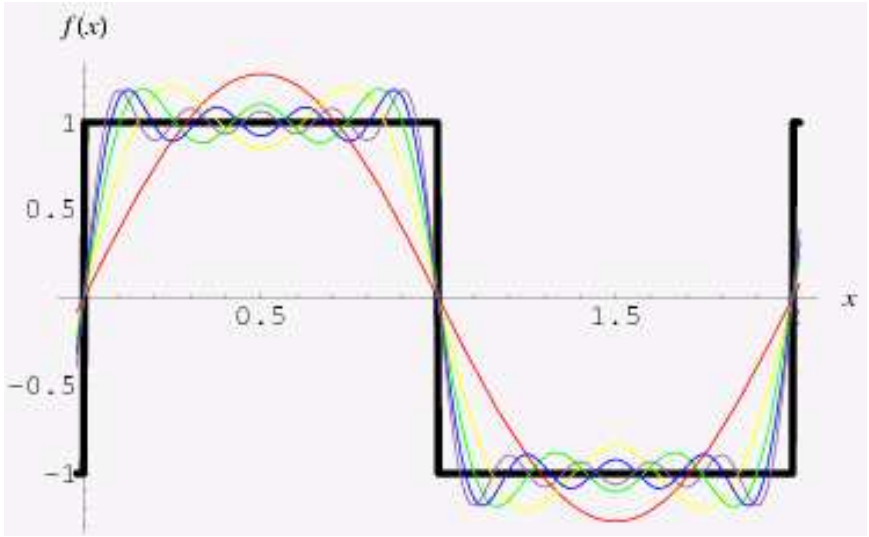
Compression: just three terms b_1, b_3, b_5 give the green curve.

Pure and Applied Mathematicians



Adrien-Marie Legendre and Joseph Fourier.
Watercolor by Julien-Leopold Boilly, c.1820.

Example of Fourier's Construction (Not So Good)



Adding up just sines with $b_n \sim 1/n$ to get a square wave.

Gibbs' phenomenon: overshoots never go away.

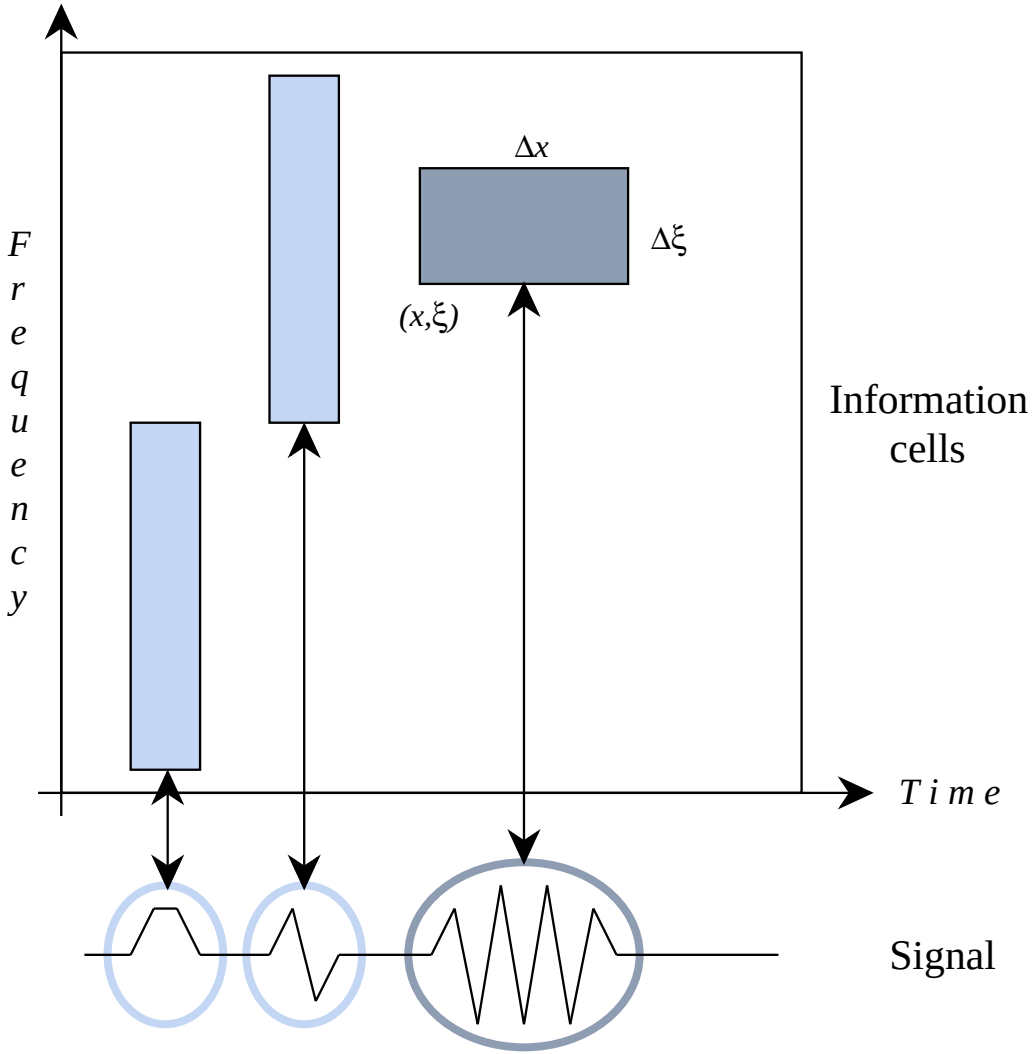
Problems with Fourier's Construction

- In fact, not all functions f equal their Fourier series.

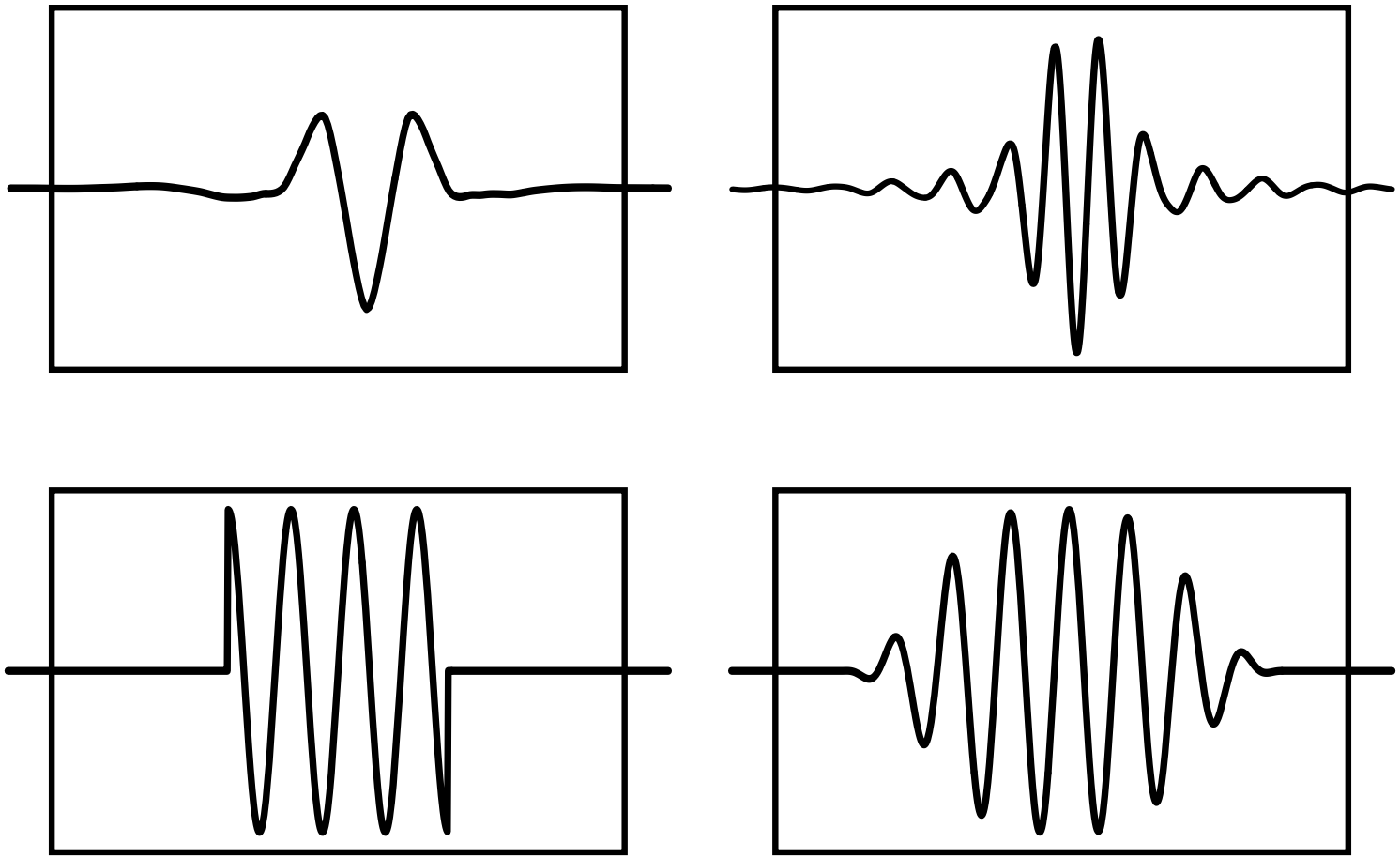
- Infinitely many numbers $\{a_n, b_n\}$ are needed to represent a given function f, and some simple functions require very many for a good approximation.

- Sines and cosines have no location and infinite duration.

Time and Frequency Content Analyzed Together



Waveforms Localized in Time and Frequency



History B.D. [Before Daubechies]

- Fourier bases (1822, Paris)
- Haar bases (1910, *Math. Annalen*)
- Gabor functions (1946, *J. IEE*)
- Balian-Low theorem (1981, *CRAS*)
- Wilson bases (1987, Cornell)

Ingrid Daubechies' Construction

Theorem 2 *Any function $f = f(t)$ may be written as a sum of wavelets $w_{jk}(t) \stackrel{\text{def}}{=} w(2^j t + k)$, multiplied by numbers c_{jk} specific to f:*

$$f(t) = \sum_{j \in \mathbf{Z}} \sum_{k \in \mathbf{Z}} c_{jk} w_{jk}(t),$$

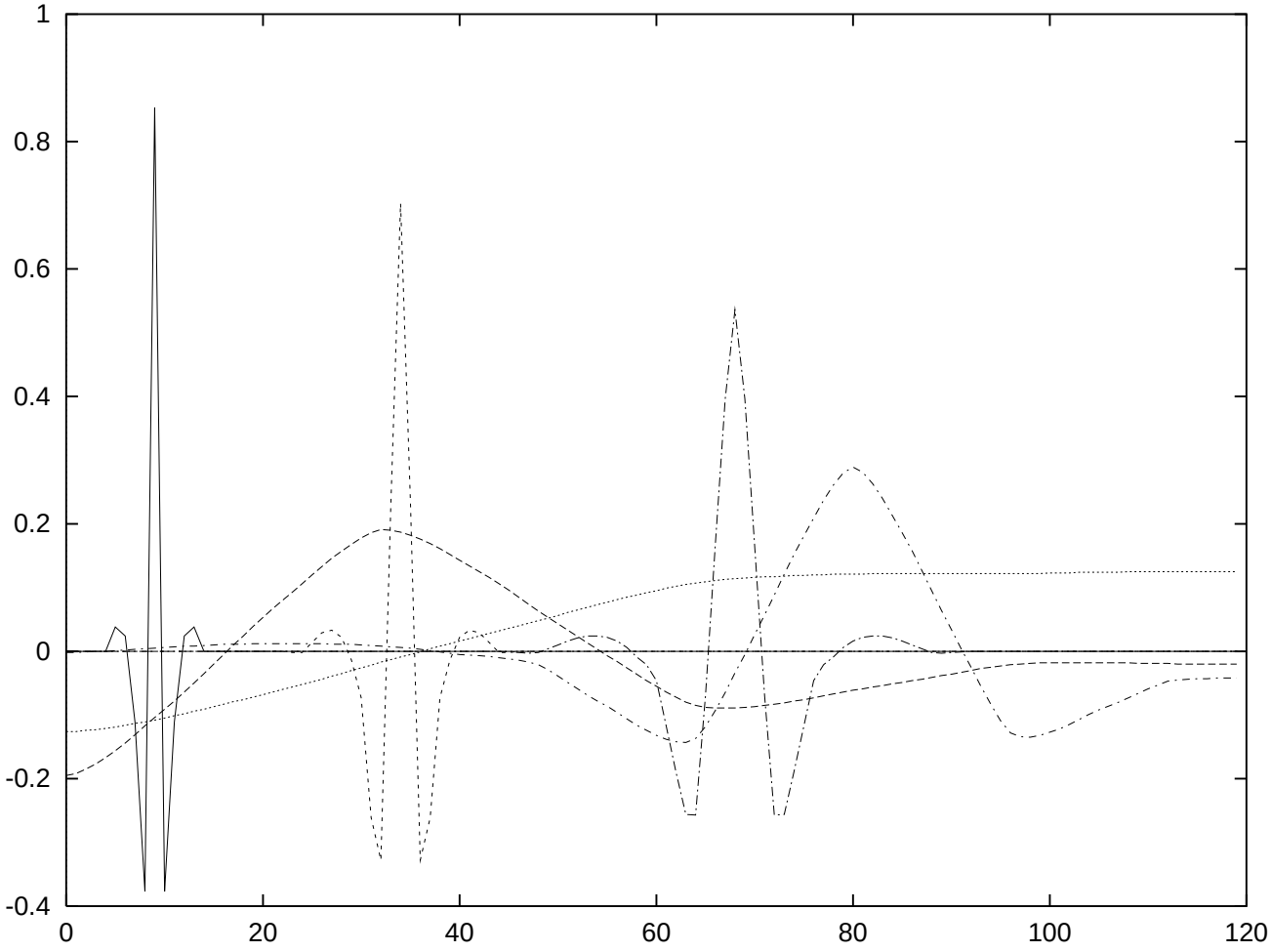
and the mother wavelet $w = w(t)$ can be chosen with these three properties:

Smoothness: *w and its first d derivatives $w', w'', \ldots, w^{(d)}$ are continuous functions.*

Compact support: *$w(t)$ is zero at all $|t| > 5d$.*

Orthogonality: *The set $\{w_{jk} : j, k \in \mathbf{Z}\}$ is an orthonormal basis.*

Some Nice Wavelets



Six dilations and translations, on an interval, of a particular mother wavelet (9,7-biorthogonal symmetric).

History A.D. [After Daubechies]

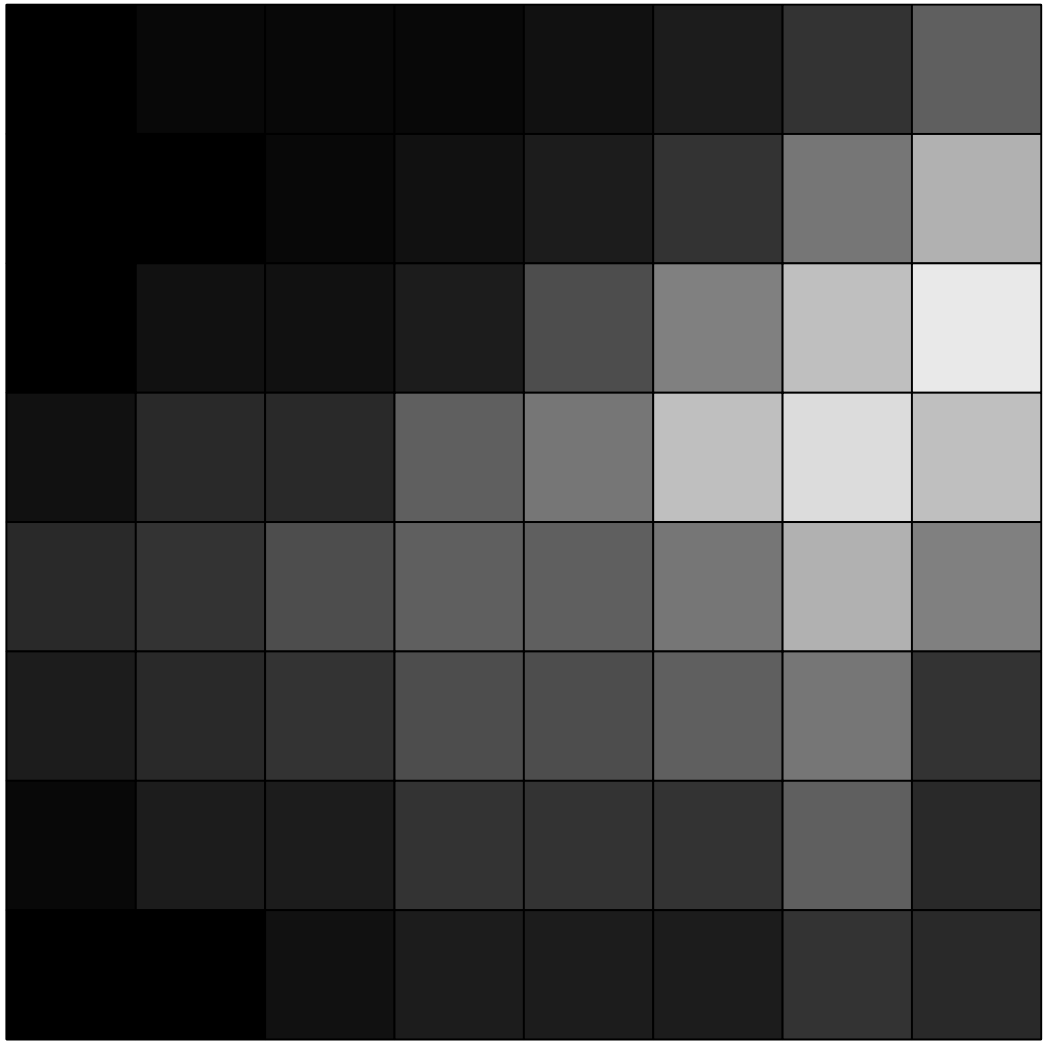
- Lapped orthogonal transforms (1990, *IEEE ASSP*)
- Biorthogonal wavelets, wavelet packets (1992, *IEEE IT*)
- WSQ fingerprint standard (1993, FBI)
- Wavelets on spheres (1995, *ACM*)
- The lifting implementation (1996, *ACHA*; 1998, *JFAA*;)
- JPEG-2000 compression (1999)

Example Images

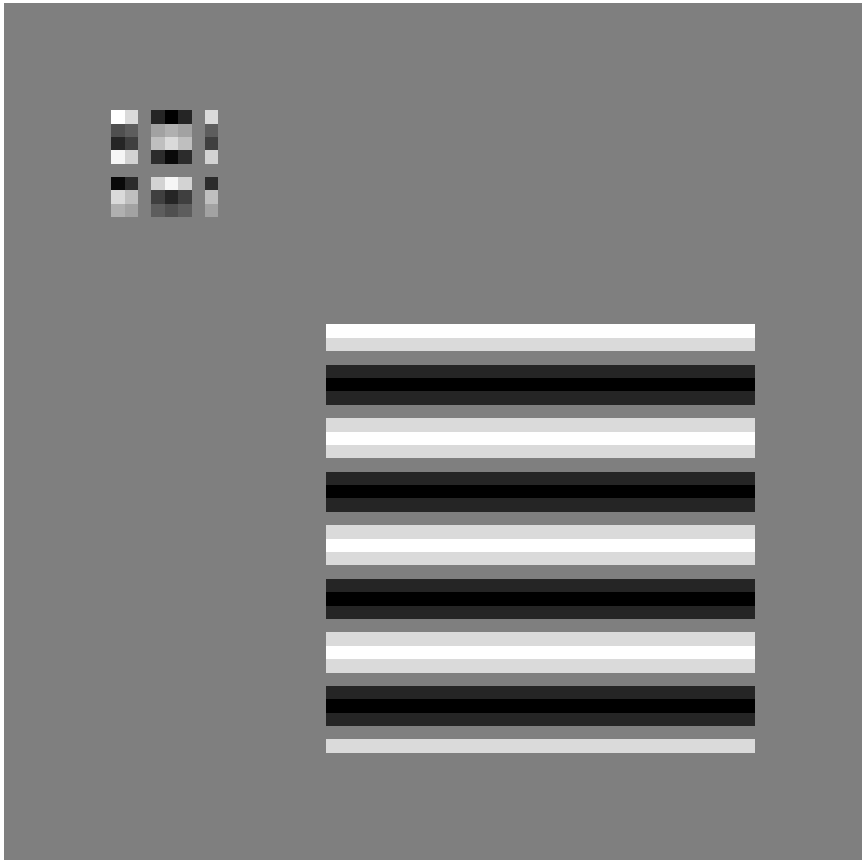


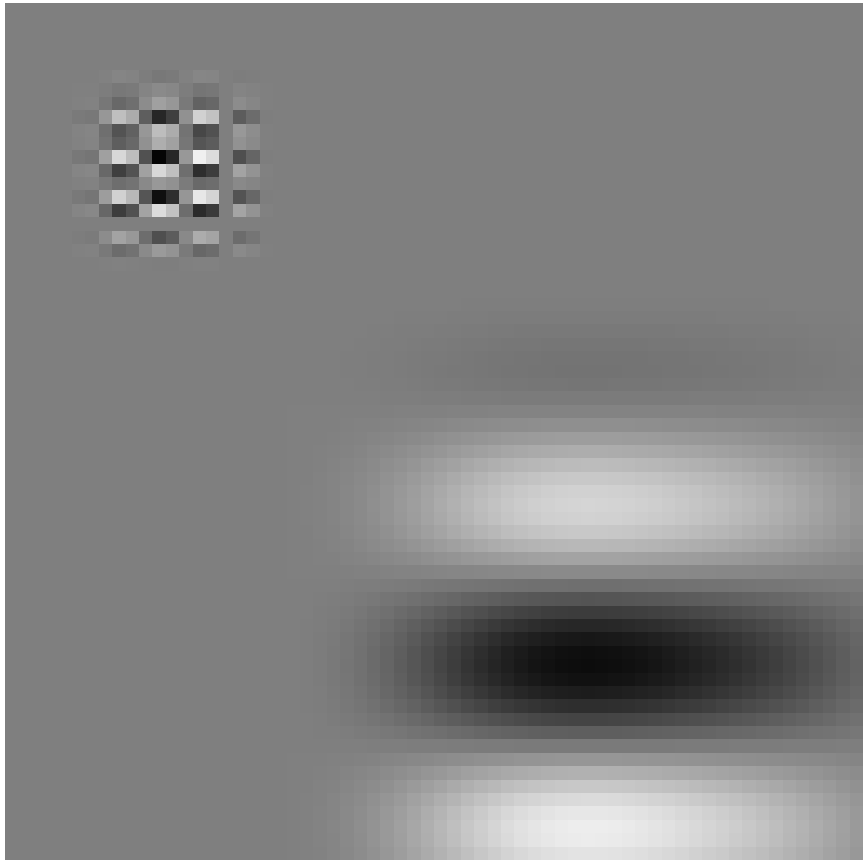


Close Up of Correlated Pixels

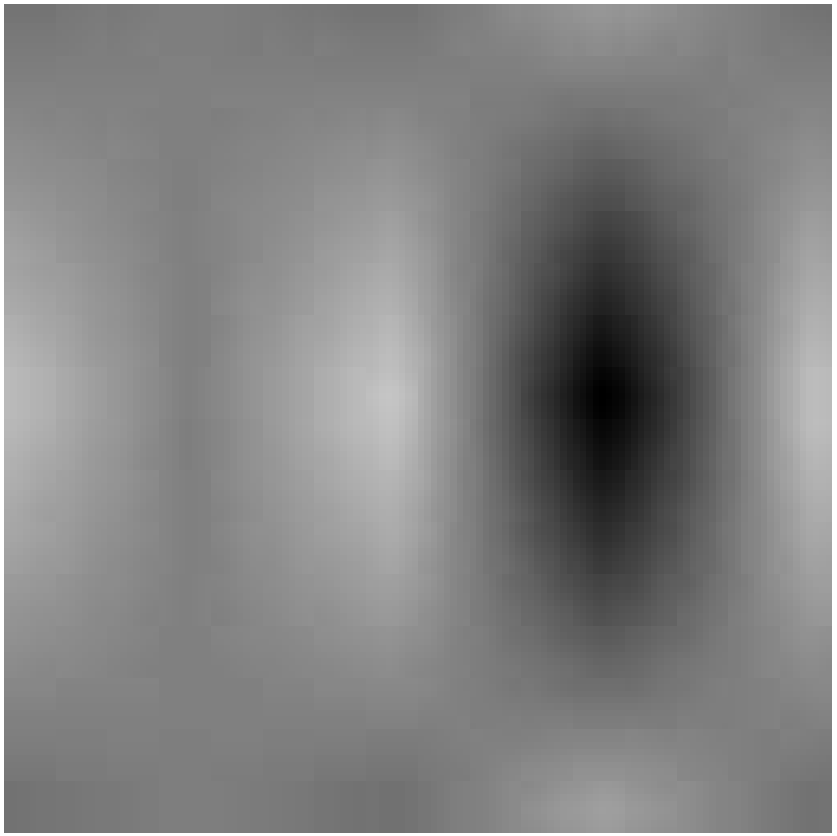


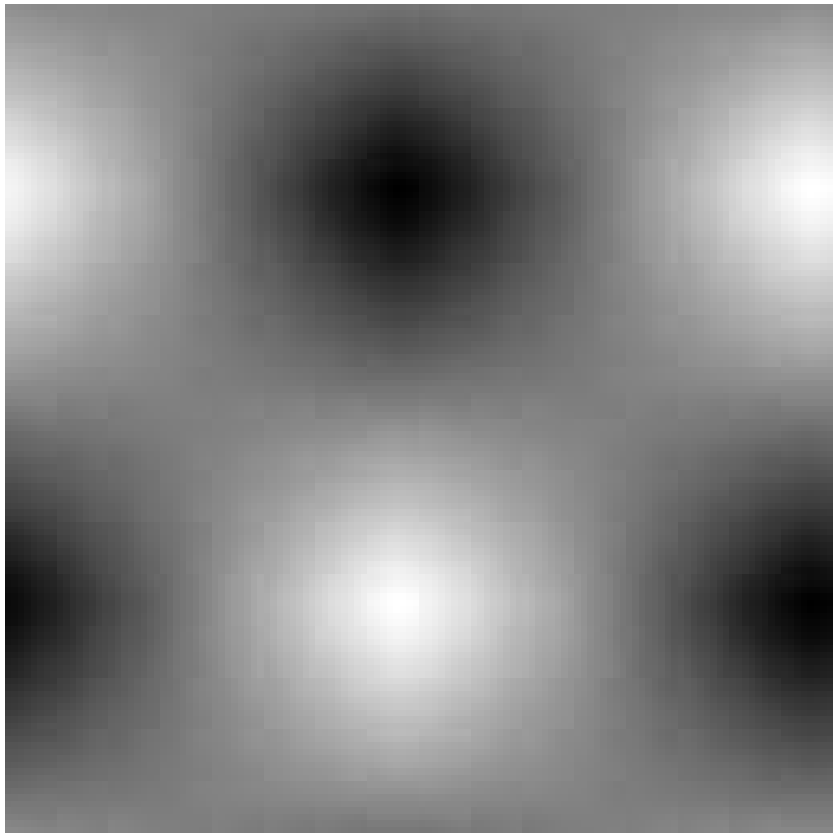
Two-Dimensional Waveforms I



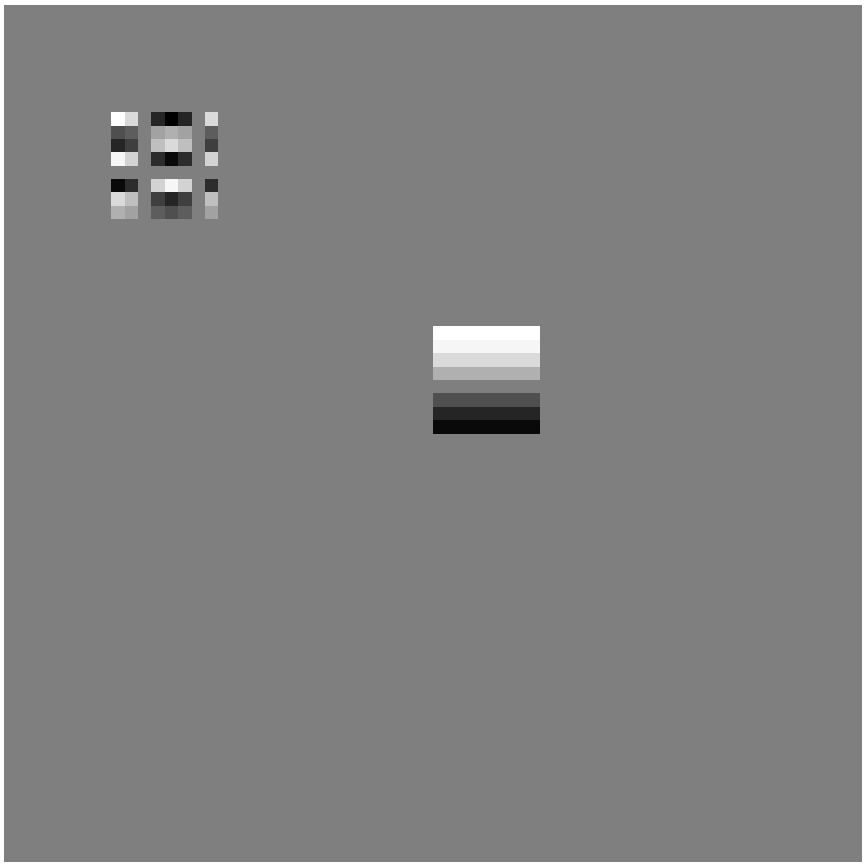


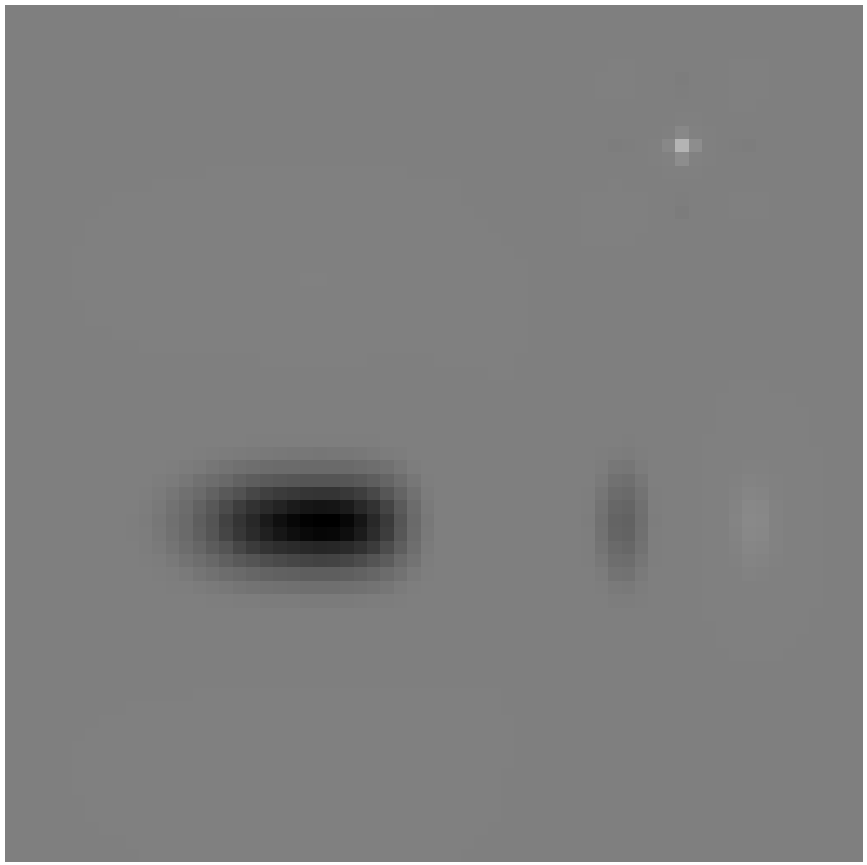
Two-Dimensional Waveforms II





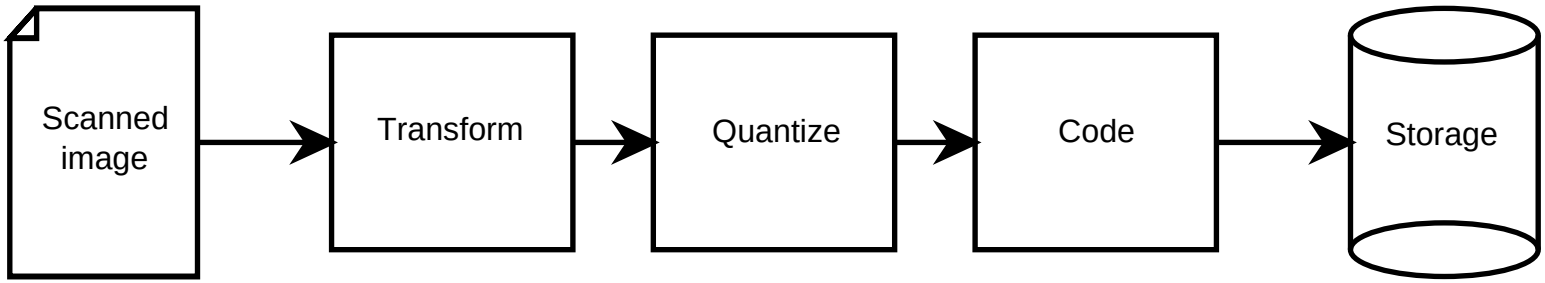
Two-Dimensional Waveforms III: JPEG vs. JPEG-2000



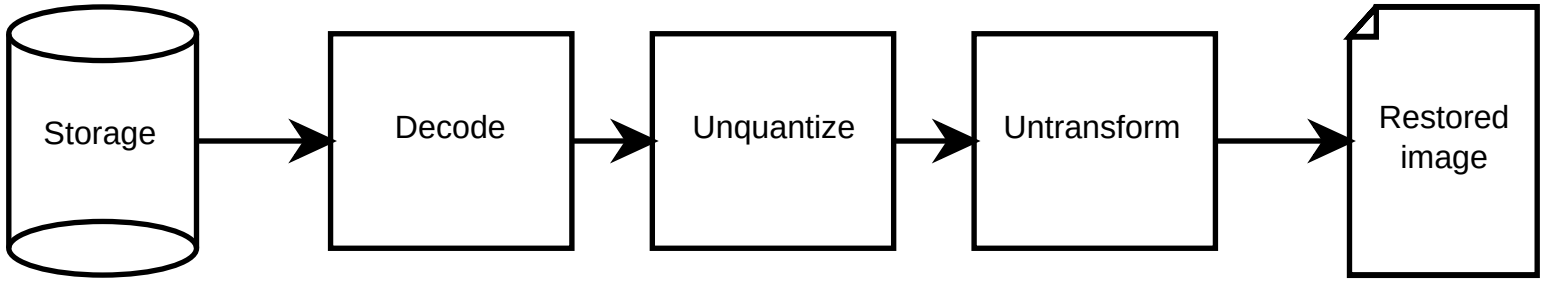


Transform Coding Image Compression

Compression:



Decompression:



Parts Description

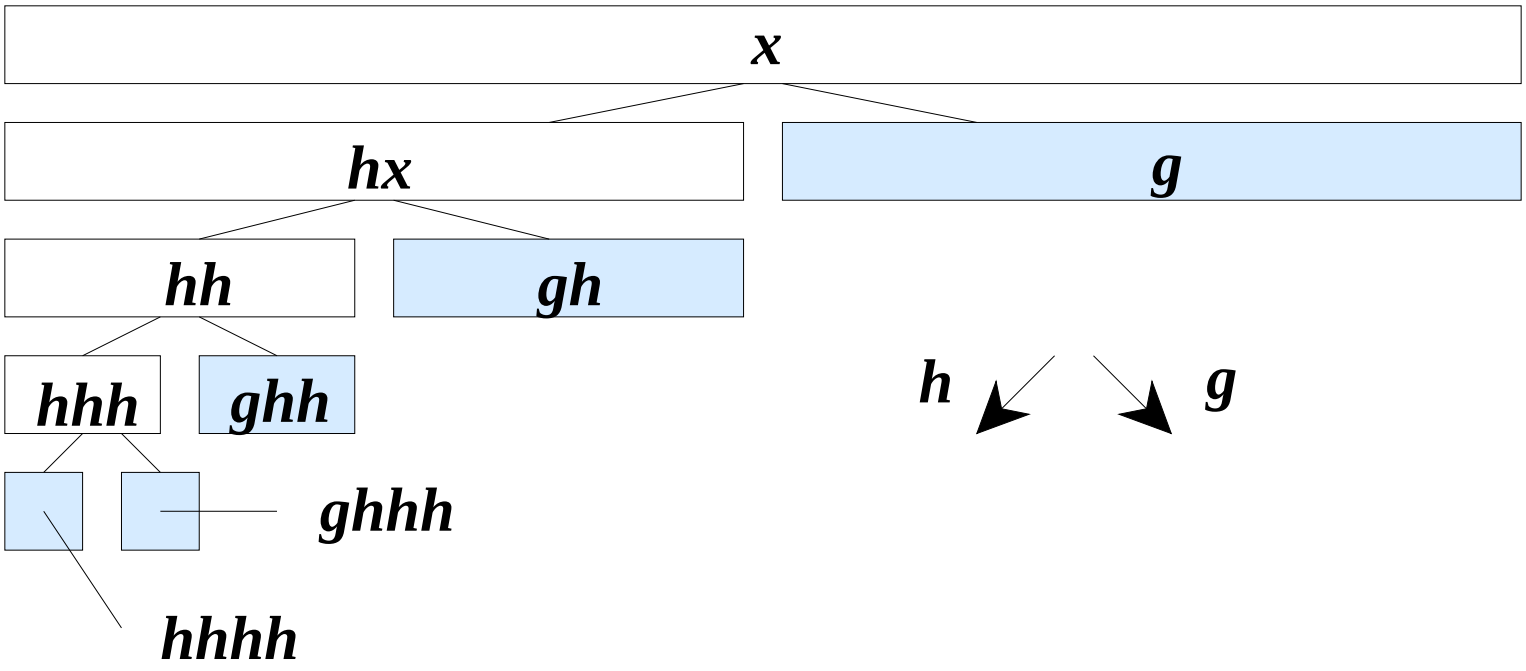
Compression:

Transform: convert pixels to amplitudes;
Quantize: round off the amplitudes to small numbers;
Code: remove redundancy from the small number sequence.

Decompression:

Decode: expand to recover the small number sequence;
Unquantize: insert an amplitude for each small number;
Untransform: recover pixels from approximate amplitudes.

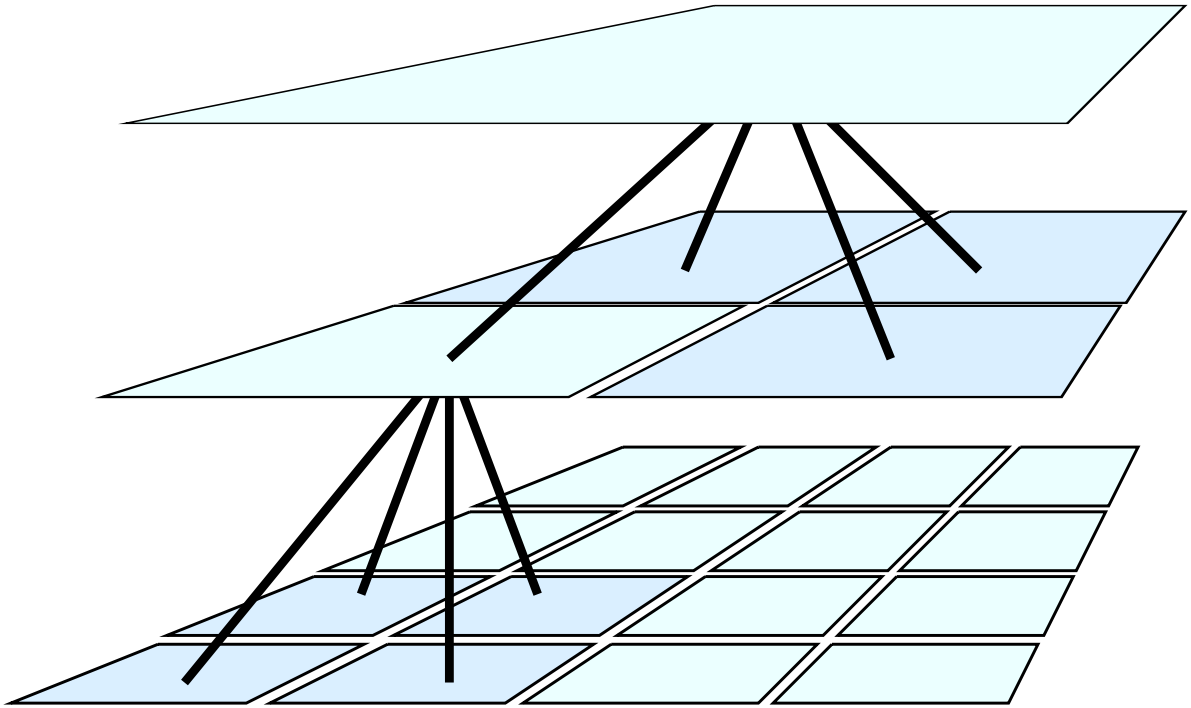
Wavelet Transform: Multiresolution Signal Splitting



Split signal x into averages hx and details gx.

Replace $x \leftarrow hx$ and repeat

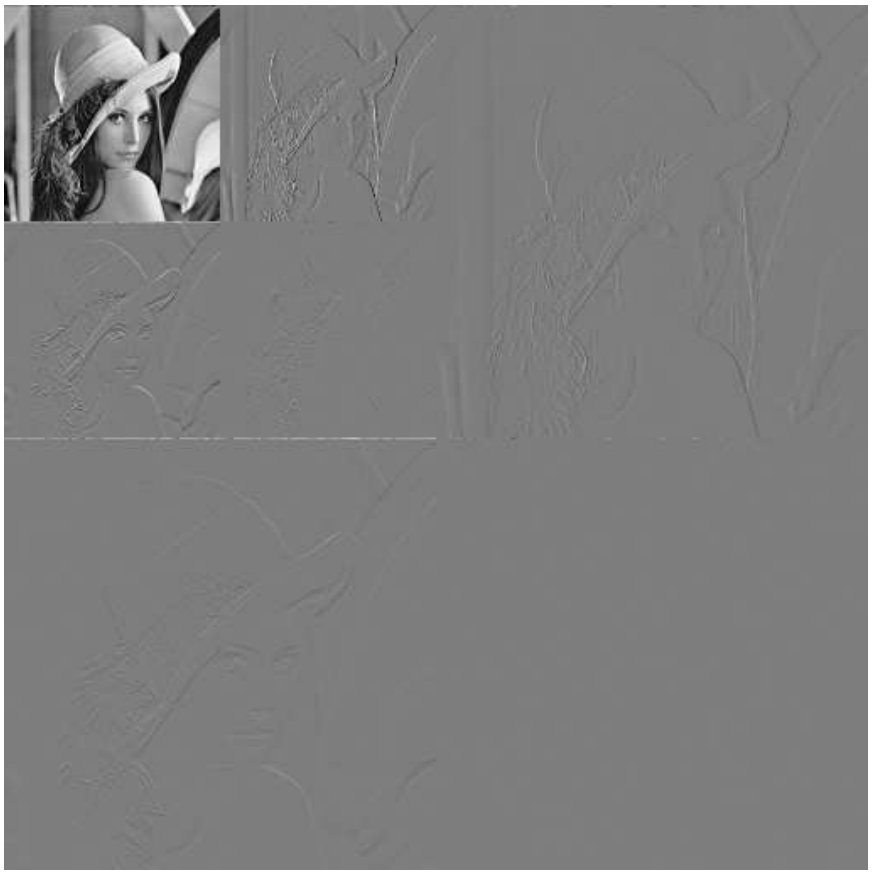
Multiresolution Image Splitting



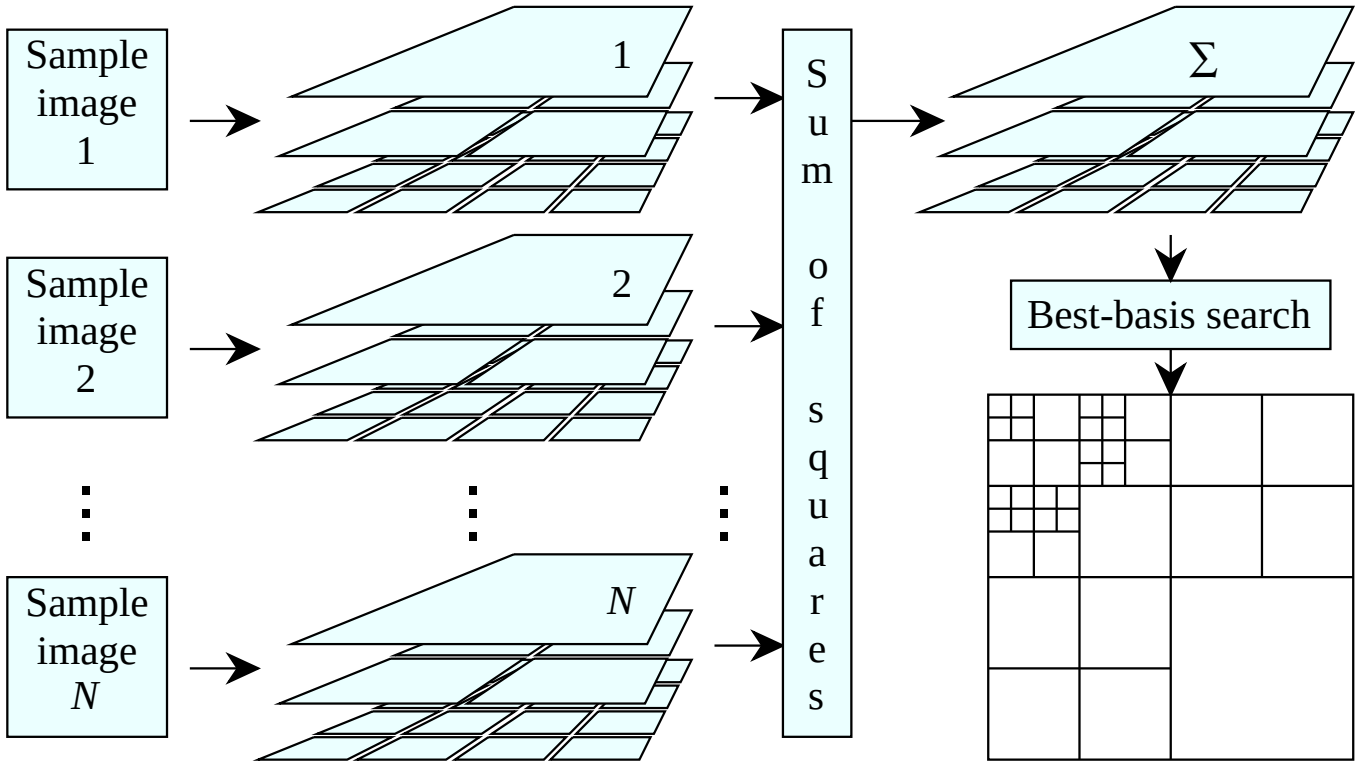
Picture (at top) becomes thumbnail (at bottom left) plus two layers of saved details (highlighted).

Storage of Multiresolution Image Data



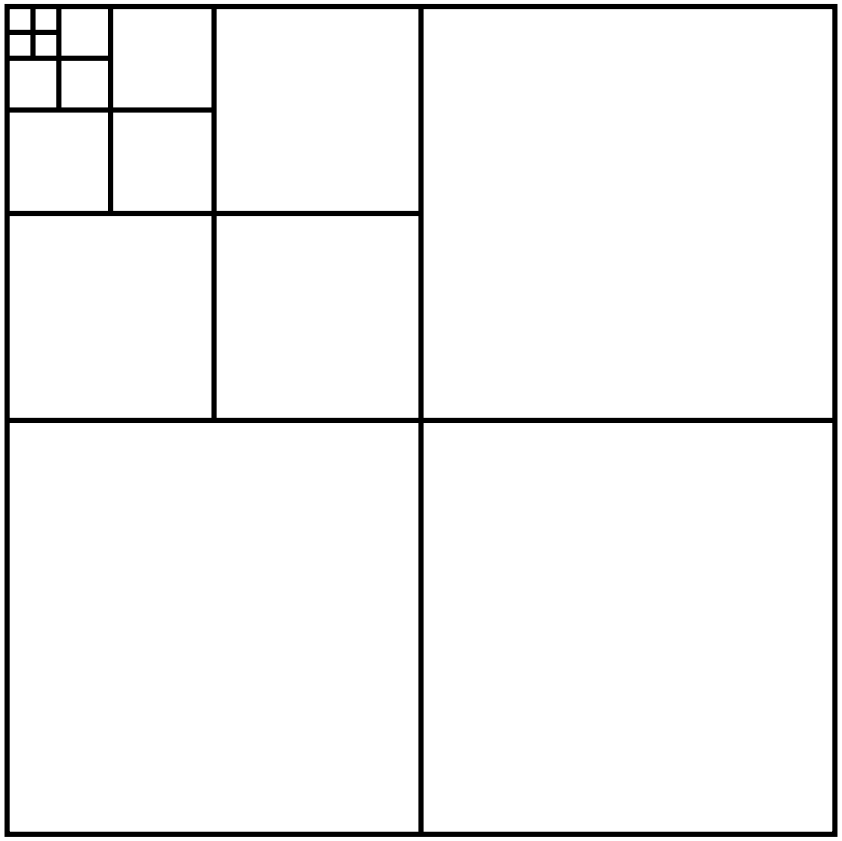


Custom Compression Algorithms



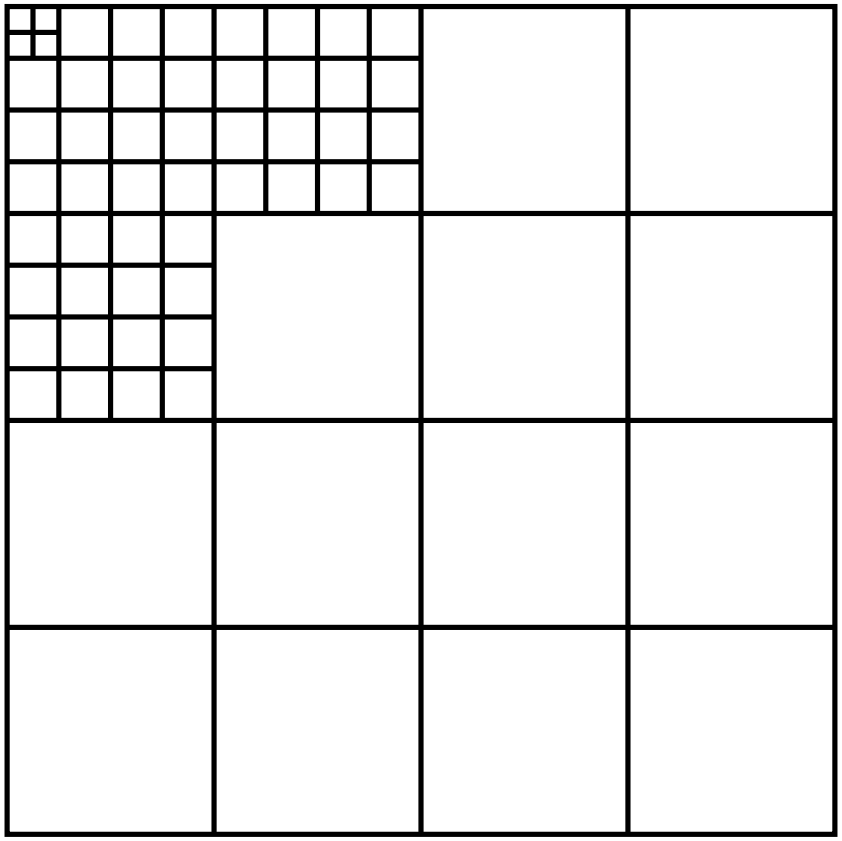
Training algorithm for a custom transform coding image compression algorithm.

Good Bases for Images I



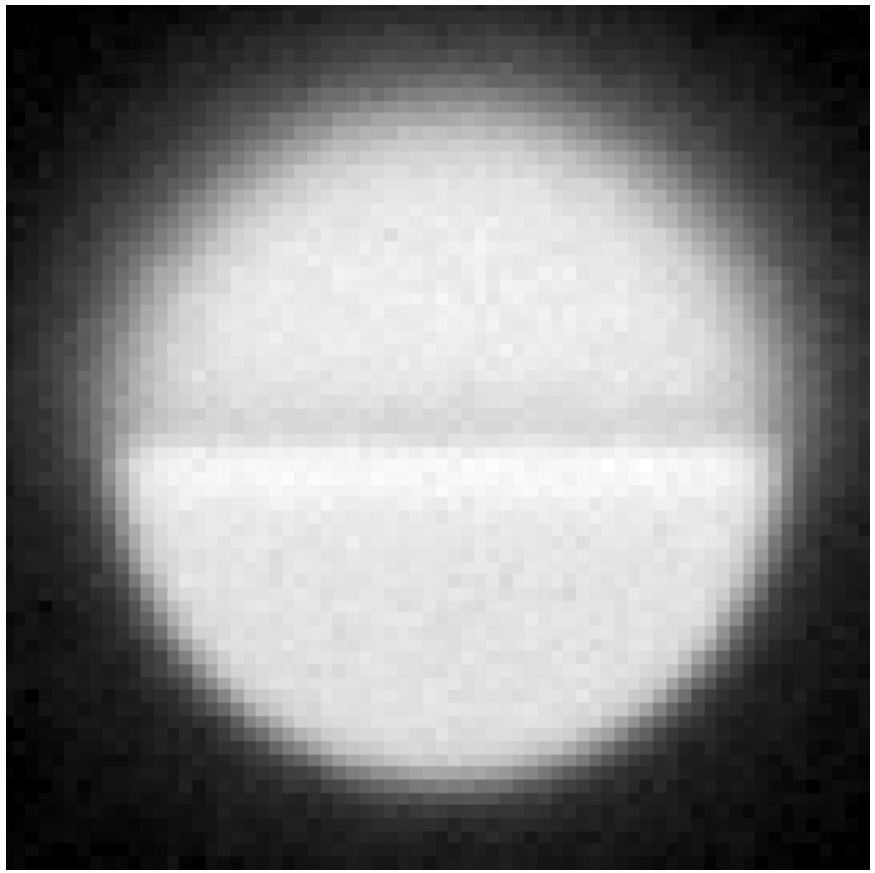
Five-level wavelet basis, used in JPEG-2000.

Good Bases for Images II



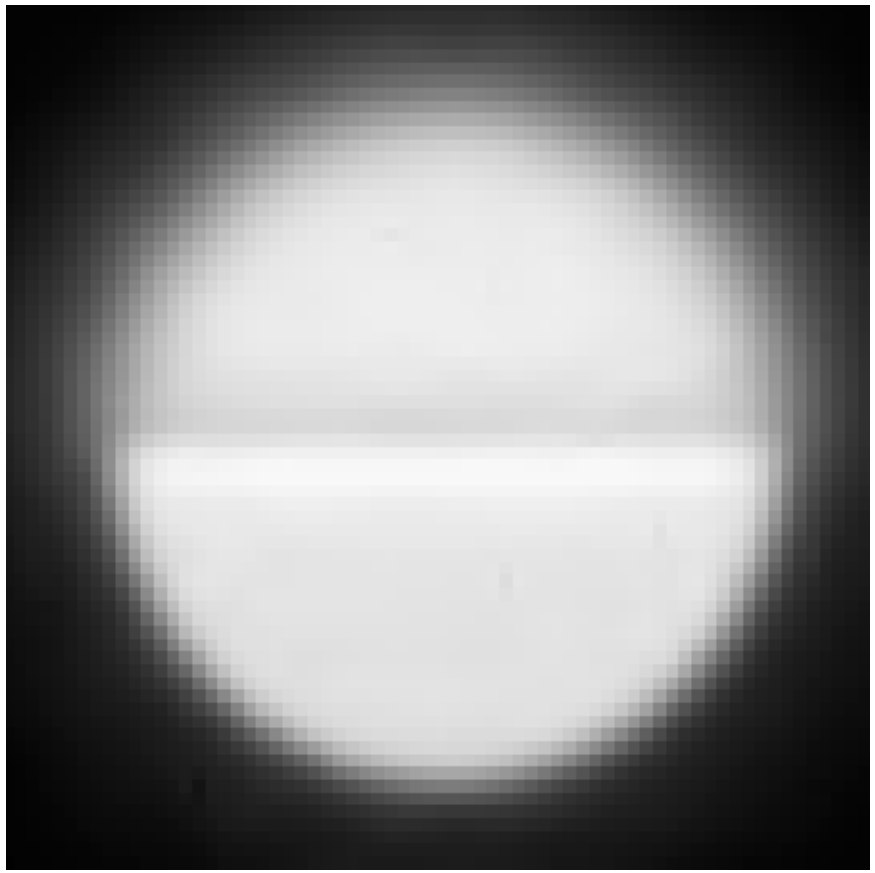
Five-level wavelet packet basis, used in WSQ.

Compression Sometimes Improves Things



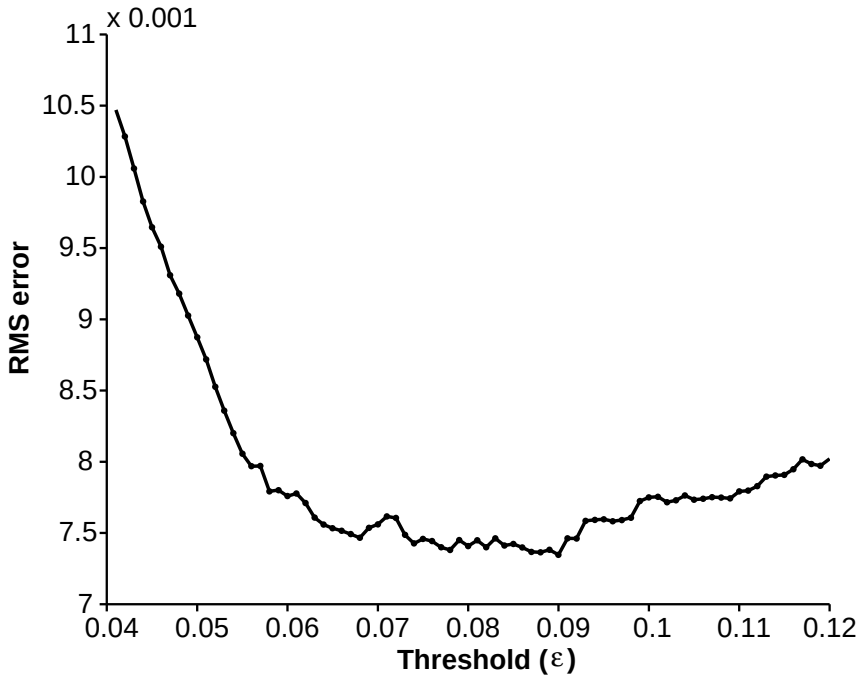
Rough Radiation Dose Approximation in 2D:
4 M particle simulation

...By Eliminating the Rough Errors



Improved Approximation in 2D:
Compressed 4 M particle simulation

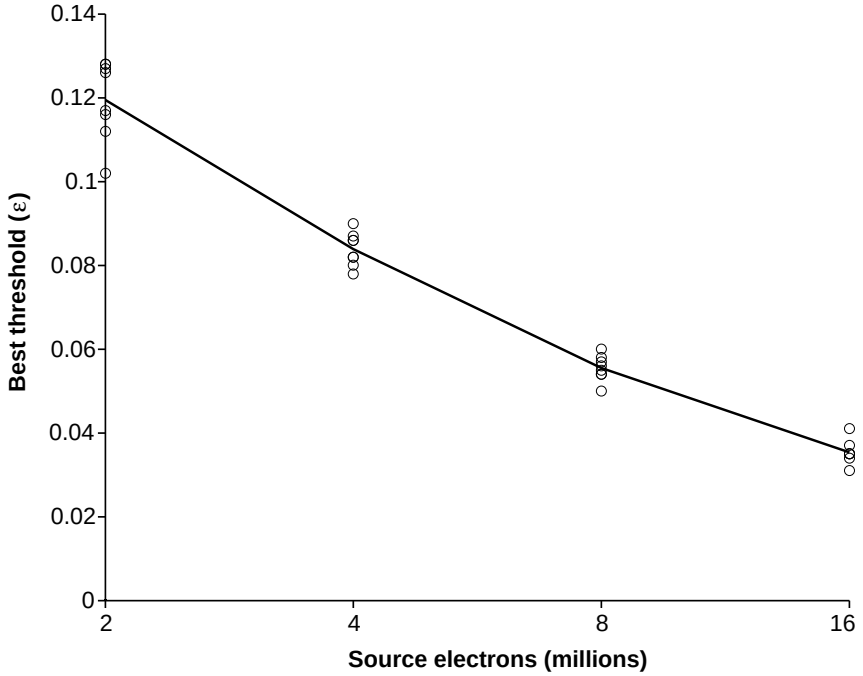
...If the Right Amount of Compression is Done



Deasy et al., Fig. 3

Reduction in RMS error by a rough approximation compressed toward a smooth target, by wavelet threshold.

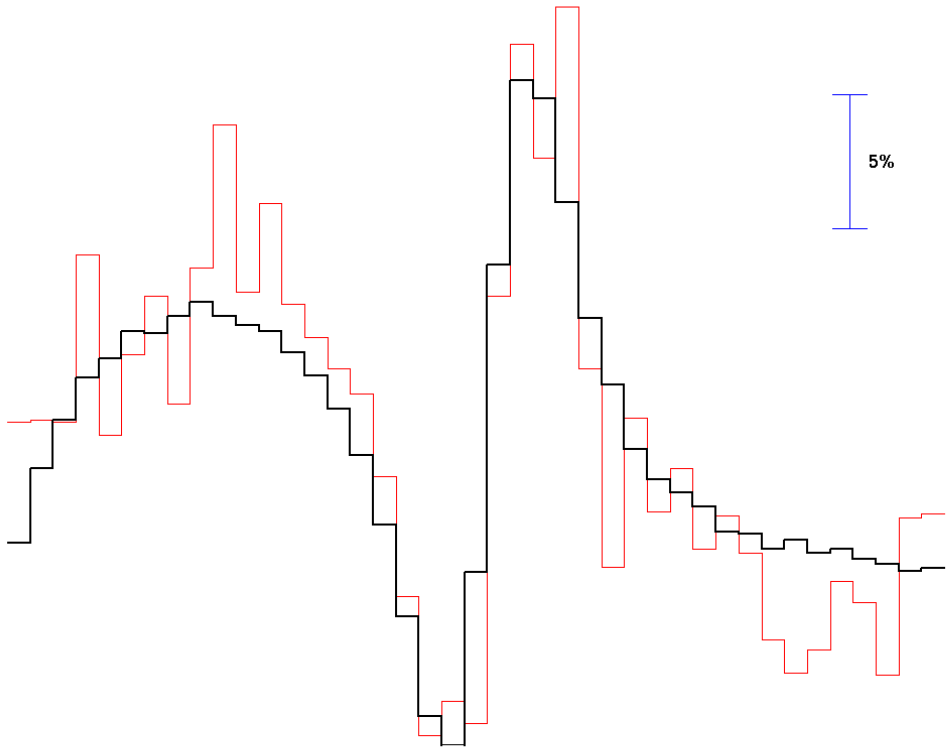
...Which, Fortunately, is Easy to Find.



Deasy, et al., Fig. 4

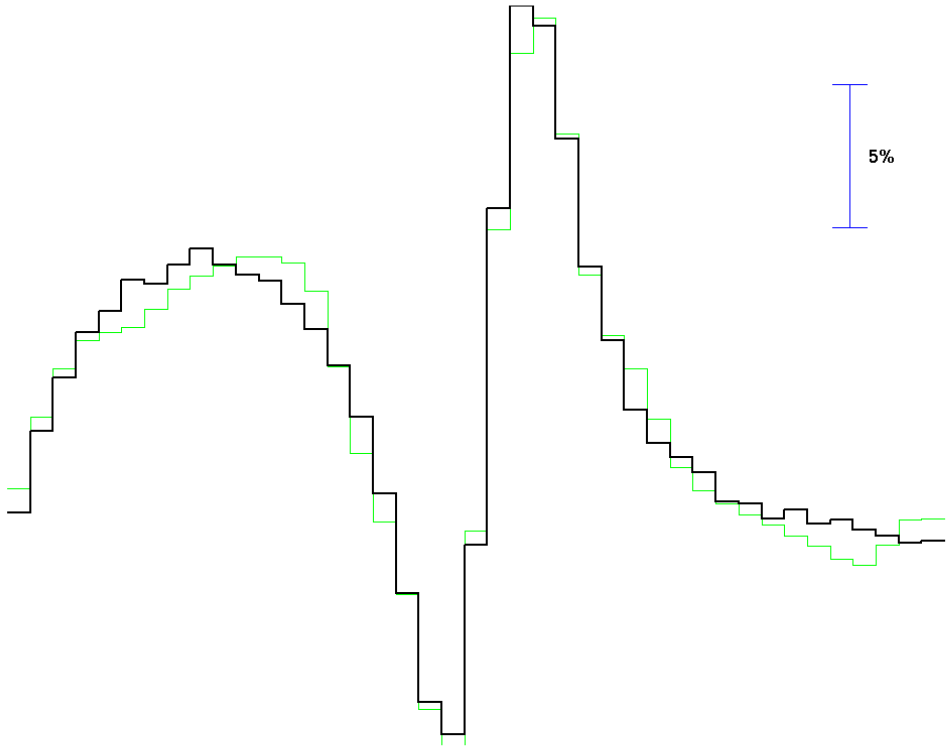
Best wavelet thresholds for compression from a rough approximation.

Example: Rough Radiation Dose Approximation – 1D



4 M particle simulation — 1D cross-section, close up.

Example: Compressed Approximation –1D



Compressed 4 M particle simulation — 1D cross-section, close up.

Some Notable Works

- Ingrid Daubechies. "Orthonormal Bases of Compactly Supported Wavelets." *Comm. Pure Appl. Math.* 41(1988),909–996.

- Ingrid Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM Press, Philadelphia, 1992.

- Albert Cohen, Ingrid Daubechies and Jean-Christophe Feauveau. "Biorthogonal Bases of Compactly Supported Wavelets" *Comm. Pure Appl. Math.* 45(1992),485–500.

- Ingrid Daubechies and Wim Sweldens. "Factoring Wavelet Transforms into Lifting Steps." *Fourier Anal. Appl.* 4:3(1998),245–267.