Putnam Practice Problems: Permutations and Combinations

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1 Permutations and Combinations

Permutations: Number of permutations (orderings) of n distinguishable objects: $n! \stackrel{\text{def}}{=} n(n-1)\cdots(2)(1)$,

Number of permutations of $n = n_1 + \cdots + n_k$ objects, with n_i objects of type i for $i = 1, \ldots, k$, and object i distinguishable from object j iff $i \neq j$: $n!/(n_1! \cdots n_k!)$.

NOTE: $0! \stackrel{\text{def}}{=} 1$.

Combinations: Number of combinations (choices) of k objects from a set of n:

$$\binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!}$$

Identities:

$$\binom{n}{k} = \binom{n}{n-k}; \qquad \binom{n}{0} = \binom{n}{n} = 1; \qquad \binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n.$$

Number of combinations of $n = n_1 + \cdots + n_k$ objects, with $n_i \geq 0$ objects of type i for $i = 1, \ldots, k$, and object i distinguishable from object j iff $i \neq j$:

$$\binom{n}{n_1, \ldots, n_k} \stackrel{\text{def}}{=} n!/(n_1! \cdots n_k!).$$

Binomials:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Multinomials:

$$(x_1 + \cdots + x_m)^n = \sum_{n_1 + \cdots + n_m = n} \binom{n}{n_1, \ldots, n_m} x_1^{n_1} \cdots x_m^{n_m},$$

where the sum is over all nonnegative values of n_i satisfying the condition $n_1 + \cdots + n_m = n$.

Derangements: A reordering of $(1, \ldots, n)$ with the property that for all k, element k is mapped to a position other than k is called a *derangement*. The number of derangements of n objects is

$$D(n) \stackrel{\text{def}}{=} n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

which is sometimes denoted by $!n$, the *subfactorial*.

Inclusion/Exclusion: Denote by $|X|$ the number of elements in a finite set X. Then $|A \cup B| = |A| + |B| - |A \cap B|$, and more generally

$$|A_1 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \cdots + (-1)^{k-1} |A_1 \cap \cdots \cap A_n|$$

2 Exercises involving permutations and combinations

2.1

Prove that $\binom{n}{k} = \binom{n}{n-k}$.

2.2

Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. (This gives *Pascal's triangle*.)

2.3

Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

2.4

Prove that $!n$ is an integer for every $n \in \mathbf{Z}^+$.

2.5

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

2.6

Prove that $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$ for any $n \geq 1$.

2.7

Prove *Vandermonde's identity:*

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}.$$

2.8

Evaluate

$$\sum_{p+q+r=5} \binom{12}{p, q, r, 7}$$

2.9

Evaluate the two expressions

$$\sum_{k=1}^{n} k\binom{n}{k}; \qquad \sum_{k=1}^{n} k^2\binom{n}{k}.$$

2.10

Use inclusion/exclusion to prove the derangements formula.