Putnam Practice Problems: Induction

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1 Mathematical induction

Let $\mathbf{Z}^+$ denote the positive integers $\{1, 2, 3, \dots\}$.

PMI: Given a list of propositions $\{P(n) : n \in \mathbf{Z}^+\}$, suppose that

A. $P(1)$ is true.

B. For all $n \geq 1$, if $P(n)$ is true, then $P(n+1)$ must be true.

Then we may conclude that $P(n)$ is true for all $n \in \mathbf{Z}^+$. We reason that if both A and B are established, then $P(1)$ is true by A, which means $P(2)$ is true by B and $P(1)$, so $P(3)$ is true by B and $P(2)$, and so on. This is the *Principle of Mathematical Induction*, or PMI.

Formally, we may write PMI as follows:

$$P(1) \wedge \big((\forall n \in \mathbf{Z}^+)P(n) \implies P(n+1)\big) \implies (\forall n \in \mathbf{Z}^+)P(n),$$

where $\wedge$ denotes AND, namely logical conjunction.

PMI is equivalent to WOP: Every nonempty subset of $\mathbf{Z}^+$ contains a least element. This is the *Well-Ordering Principle*, or WOP. PMI implies WOP, and WOP implies PMI.

Strong induction: We may assume $P(1), \dots, P(n)$ when proving $P(n+1)$. Suppose that

A. $P(1)$ is true.

B. For all $n \in \mathbf{Z}^+$, if $P(1), \dots, P(n)$ are true, then $P(n+1)$ is true.

Then we may conclude that $P(n)$ is true for all $n \in \mathbf{Z}^+$. This is known as *strong induction*, and it is equivalent to ordinary induction but with more things spelled out in the proof.

Formally,

$$P(1) \wedge \big[(\forall n \in \mathbf{Z}^+)\left((\forall k \leq n)P(k)\right) \implies P(n+1)\big] \implies (\forall n \in \mathbf{Z}^+)P(n).$$

Strong induction from some other initial case: The first true case may be $P(m)$ with $m \geq 1$, and we may have to check $k \geq 0$ additional initial cases by hand. Suppose that

A. $P(m), \dots, P(m+k)$ are all true.

B. For all $n \in \mathbf{Z}^+$ with $n \geq m+k$, if $P(m), \dots, P(n)$ are true, then $P(n+1)$ is true.

Then we may conclude that $P(n)$ is true for all $n \in \mathbf{Z}^+$ with $n \geq m$. Again, this is equivalent to PMI with renumbering.

Formally

$$P(m) \wedge \dots \wedge P(m+k) \wedge [(\forall n \geq m+k)P(m) \wedge \dots \wedge P(n) \implies P(n+1)] \implies (\forall n \geq m)P(n).$$

2 Exercises involving induction

2.1

Prove that $n! > 2^n$ for all $n \geq 4$.

2.2

Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

2.3

Let F_n be the nth Fibonacci number, defined by $F_1 = F_2 = 1$ with $F_{n+2} = F_{n+1} + F_n$ for all integers $n \geq 1$. Prove that $F_n < 2^n$ for all $n \geq 1$.

2.4

Let F_n be the nth Fibonacci number. Prove that F_{5n} is divisible by 5 for every $n \geq 1$.

2.5

Suppose r is a number such that $r + 1/r$ is an integer. Prove that $r^n + 1/r^n$ is an integer for all $n \geq 1$.

2.6

Find the maximum number $R(n)$ of regions into which the plane can be divided using n lines.

2.7

Prove that if $n \geq 44$, then n cents of postage can be made up using only 5 cent and 12 cent stamps.

2.8

Call an integer "squarefull" if every prime in its prime factorization occurs to a power of 2 or more. Prove that there are infinitely many consecutive squarefull integers.

2.9

Prove that for every $n \geq 2$, the expansion of $(1 + x + x^2)^n$ contains at least one even coefficient.

2.10

Define the *Ulam numbers* $\{u_n : n \in \mathbf{Z}^+\}$ recursively by $u_1 = 1$, $u_2 = 2$, and for $n \geq 2$ let u_{n+1} be the least integer greater than u_n that can be written *in exactly one way* as a sum of two distinct Ulam numbers in $\{u_1, \ldots, u_n\}$. Prove that there are infinitely many Ulam numbers.