Putnam Practice Problems: Inequalities

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1 Some useful inequalities

1+x vs. 1-x: For real $x \in (-1,1)$ we have $1-x \le 1/(1+x)$ and $1+x \le 1/(1-x)$, with equality iff $x=0$.

Exp vs. linear vs. Log: For all x we have $1+x \le e^x$ and $\log(1+x) \le x$, with equality iff $x=0$.

Bernoulli: For all $x > -1$,

$$\begin{array}{rcl} r \in (-\infty,0] \cup [1,\infty) & \implies & (1+x)^r \ge 1+rx; \\ r \in (0,1) & \implies & (1+x)^r \le 1+rx. \end{array}$$

Equality holds iff $r=0$ or $r=1$ or $x=0$.

Shannon: For positive real numbers $p_1,\ldots,p_n$ and $q_1,\ldots,q_n$ satisfying $p_1+\cdots+p_n = q_1+\cdots+q_n = 1$, we have

$$\sum_{k=1}^n p_k \log p_k \ge \sum_{k=1}^n q_k \log p_k$$

with equality iff $p_k = q_k$ for all $k = 1,\ldots,n$.

Cauchy–Schwarz: Suppose that $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ are real numbers. Then

$$\left(\sum_{k=1}^n a_k b_k\right)^2 \le \left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n b_k^2\right)$$

with equality iff there exist x,y not both zero such that $a_k x + b_k y = 0$ for all $k = 1,\ldots,n$.

AGHM: Suppose that $a_1,\ldots,a_n$ are positive real numbers. Then

$$\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}} \le (a_1\cdots a_n)^{1/n} \le \frac{a_1+\cdots+a_n}{n} \qquad \text{equiv.} \quad \left(\frac{1}{n}\sum_{k=1}^n \frac{1}{a_k}\right)^{-1} \le \left(\prod_{k=1}^n a_k\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_{k=1}^n a_k,$$

with equality (in each comparison) iff $a_1 = \cdots = a_n$.

Jensen: Suppose that $\phi : \mathbf{R} \to \mathbf{R}$ is twice continuously differentiable and satisfies $\phi''(x) \ge 0$ for all $x \in [a,b]$. Then for any $x_1,\ldots,x_n \in [a,b]$ we have

$$\phi\left(\frac{1}{n}\sum_{k=1}^n x_k\right) \le \frac{1}{n}\sum_{k=1}^n \phi(x_k),$$

with equality iff $x_1 = \cdots = x_n$ or $\phi''(x) = 0$ for all $x \in [a,b]$.

2 Exercises involving inequalities

2.1

Determine which number is larger: $\sqrt{19} + \sqrt{99}$, or $\sqrt{20} + \sqrt{98}$.

2.2

Determine which number is larger: $2000!$, or 1000^{2000}.

2.3

Determine which number is largest: 1999^{1999}, 2000^{1998}, or 1998^{2000}.

2.4

Find the minimum surface area of a rectangular solid that has volume V. Do not use calculus.

2.5

What is the largest possible value of $(\sin A_1) \cdots (\sin A_n)$, given that $(\tan A_1) \cdots (\tan A_n) = 1$?

2.6

Suppose that a, b, c are nonnegative real numbers and $(a+1)(b+1)(c+1) \leq 8$. Prove that $abc \leq 1$.

2.7

Suppose that $x_1, \ldots, x_n > 0$ and $x_1 + \cdots + x_n = 1$. Prove that $\displaystyle\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}$.

2.8

Suppose that $x_1, \ldots, x_n > 0$ and $x_1 \cdots x_n = 1$. Prove that $\displaystyle\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} x_i^2$.

2.9

Prove that for all $a > b > 0$ we have $\sqrt{ab} < \dfrac{a-b}{\log a - \log b} < \dfrac{a+b}{2}$

2.10

Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where $a_1, \ldots, a_n$ are real numbers. Given that $|f(x)| \leq |\sin x|$ for all x, prove that $|a_1 + 2a_2 + \cdots n a_n| \leq 1$.

2.11

Suppose $a_1, a_2, \ldots$ are positive real numbers, and that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges. Prove that the following series is also convergent:

$$\sum_{n=1}^{\infty} \frac{n^2\, a_n}{(a_1 + \cdots + a_n)^2}.$$