Math 5051
Measure Theory and Functional Analysis I
Homework Assignment 6

Prof. Wickerhauser

Due **Monday, December 14th, 2015**

Please do Exercises 2, 3, 6*, 8, 9, 10, 14*, 16*, 17, 18, 19*, 20.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. These exercises originate from that source.

1. Find the Fourier transform of $\chi_{[a,b]}$. In particular, find the Fourier transform of $\chi_{[-n,n]}$.

2. Find a real-valued function $f \in L^1$ such that $\hat{f} \notin L^1$.

3. Prove that $\|\hat{f}\|_\infty \le \|f\|_1$, and find a real-valued function $f \in L^1$ such that $\|\hat{f}\|_\infty < \|f\|_1$.

4. Show that if $f \in L^1$ and f is everywhere strictly positive, then $|\hat{f}(y)| < \hat{f}(0)$ for $y \neq 0$.

5. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is integrable and has integrable partial derivatives $f_j \stackrel{\text{def}}{=} \partial f/\partial x_j$ for $j = 1, \ldots, n$. Prove that the Fourier transform of f_j is given by $\hat{f}_j(u) = -iu_j\hat{f}(u)$.

6. Let $\mathcal{S}$ be the class of real-valued functions f on $\mathbf{R}$ such that for every $k \ge 0$ and $m \ge 0$, $|x|^m|f^{(k)}(x)| \to 0$ as $|x| \to \infty$, where $f^{(k)}$ is the kth derivative of f when $k \ge 1$, and $f^{(0)} \stackrel{\text{def}}{=} f$. (This collection $\mathcal{S}$ is called the *Schwartz class.*) Prove that if $f \in \mathcal{S}$, then $\hat{f} \in \mathcal{S}$.

7. The Fourier transform of a finite signed measure μ on $\mathbf{R}^n$ is defined by

$$\hat{\mu} \stackrel{\text{def}}{=} \int_{\mathbf{R}^n} e^{iu\cdot x}\,\mu(dx).$$

   Prove that if μ and ν are two finite signed measures on $\mathbf{R}^n$ (with respect to the completion of $\mathcal{L} \times \cdots \times \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue σ-algebra on $\mathbf{R}$) such that $\hat{\mu}(u) = \hat{\nu}(u)$ for all $u \in \mathbf{R}^n$, then $\mu = \nu$.

8. If $f \in L^2(\mathbf{R})$ is real-valued and continuously differentiable on $\mathbf{R}$, prove that

$$\left(\int |f|^2\,dx\right)^2 \le 4\left(\int |xf(x)|^2\,dx\right)\left(\int |f'|^2\,dx\right).$$

9. Prove *Heisenberg's inequality*: there exists $c > 0$ such that if $a, b \in \mathbf{R}$ and $f \in L^2$, then

$$\left(\int (x-a)^2|f(x)|^2\,dx\right)\left(\int (x-b)^2|\hat{f}(u)|^2\,du\right) \ge c\left(\int |f(x)|^2\,dx\right)^2.$$

Find the best constant c. (Hint: use Exercise 8.)

10. Suppose F is a closed subset of $[0,1]$ and define

$$L(f) = \int_0^1 f\chi_F\,dx$$

for real-valued continuous functions f on $[0,1]$. Prove that if μ is the measure whose existence is given by the Riesz representation theorem, then $\mu(A) = m(A \cap F)$, where m is Lebesgue measure.

11. Suppose X is a compact metric space and μ is a finite regular measure on $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra. Prove that if f is a real-valued measurable function and $\epsilon > 0$, then there exists a closed set F such that $\mu(F^c) < \epsilon$ and the restriction of f to F is a continuous function on F.

12. Let $C^1([0,1])$ be the set of functions $f : [0,1] \to \mathbf{R}$ such that f' exists and is continuous on $[0,1]$. Suppose L is a linear functional on $C^1([0,1])$ such that

$$|L(f)| \le c_1\|f'\| + c_0\|f\|$$

for all $f \in C^1([0,1])$, where c_0 and c_1 are positive constants and the norm is the supremum norm. Show that there exists a signed measure μ on the Borel subsets of $[0,1]$ and a constant K such that

$$\left(\forall f \in C^1([0,1])\right)\ L(f) = \int f'\,d\mu + Kf(0).$$

13. Suppose X and Y are compact metric spaces and $F : X \to Y$ is a continuous surjection. If ν is a finite measure on the Borel sets of Y, prove that there exists a measure μ on the Borel sets of X such that

$$\int_Y f\,d\nu = \int_X f \circ F\,d\mu$$

for all f that are continuous on Y.

14. Let X be a compact metric space. Prove that $\mathcal{C}(X)$ has a countable dense subset.

15. Let X be a compact metric space and let $\mathcal{B}$ be the Borel σ-algebra on X. Let μ_n be a sequence of finite measures on $(X, \mathcal{B})$ and let μ be another finite measure on $(X, \mathcal{B})$. Suppose $\mu_n(X) \to \mu(X)$. Prove that the following are equivalent:

    (1) $\int f\, d\mu_n \to \int f\, d\mu$ whenever f is a continuous real-valued function on X;

    (2) $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ for all closed subsets F of X.

    (3) $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ for all open subsets G of X.

    (4) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ whenever A is a Borel subset of X such that $\mu(\partial A) = 0$, where $\partial A \stackrel{\text{def}}{=} \bar{A} - A^o$ is the boundary of A.

16. Let X be a compact metric space and let $\mathcal{B}$ be the Borel σ-algebra on X. Let μ_n be a sequence of finite (positive) measures on $(X, \mathcal{B})$ and suppose $\sup_n \mu_n(X) < \infty$.

    a. Prove that if $f \in \mathcal{C}(X)$, there is a subsequence $\{n_j\}$ such that $\int f\, d\mu_{n_j}$ converges.

    b. Let A be a countable dense subset of $\mathcal{C}(X)$. Prove that there is a subsequence $\{n_j\}$ such that $\int f\, d\mu_{n_j}$ converges for all $f \in A$.

    c. With $\{n_j\}$ as in part b, prove that $\int f\, d\mu_n$ converges for all $f \in \mathcal{C}(X)$.

    d. With $\{n_j\}$ as in part b, let $L(f) \stackrel{\text{def}}{=} \lim_{j\to\infty} \int f\, d\mu_{n_j}$. Prove that $L(f)$ is a positive linear functional on $\mathcal{C}(X)$. Conclude that there exists a measure μ such that

    $$\int f\, d\mu_{n_j} \to \int f\, d\mu$$

    for all f in $\mathcal{C}(X)$.

17. Prove that if X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra on X, and μ and ν are two finite positive measures on $(X, \mathcal{B})$ such that

    $$\int f\, d\mu = \int f\, d\nu$$

    for all f in $\mathcal{C}(X)$, then $\mu = \nu$.

18. Prove that if X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra on X, and μ and ν are two finite signed measures on $(X, \mathcal{B})$ such that

    $$\int f\, d\mu = \int f\, d\nu$$

    for all f in $\mathcal{C}(X)$, then $\mu = \nu$.

19. State and prove a version of the Riesz representation theorem for complex measures.

20. Prove that if X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra on X, and μ is a complex measure on $(X, \mathcal{B})$, then the total variation of μ as defined in HW 4, Exercise 30 (which is Exercise 13.4 on textbook p.104), equals

$$\sup\left\{\left|\int f\,d\mu\right| : f \in \mathcal{C}(X),\ |f| \leq 1\right\}.$$