Math 5051
Measure Theory and Functional Analysis I
Homework Assignment 5

Prof. Wickerhauser

Due Monday, November 16th, 2015

Please do Exercises 3, 6*, 11, 13, 18, 20*, 21, 30*, 31, 32, 35, 49*.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. These exercises originate from that source.

1. Show that if f and g are absolutely continuous on an interval $[a,b]$, then so is their product fg.

2. Suppose that f and g are absolutely continuous functions on an interval $[a,b]$. Prove the *integration by parts* formula:

$$f(b)g(b) - f(a)g(a) = \int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx.$$

3. Prove that if f is integrable and real-valued, $a \in \mathbf{R}$, and

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(y)\,dy,$$

then F is of bounded variation and is absolutely continuous.

4. Suppose that f is a real-valued continuous function on $[0,1]$ and that $\epsilon > 0$. Prove that there exists a continuous function g such that $g'(x)$ exists and equals 0 for a.e. x and

$$\sup_{x\in[0,1]} |f(x) - g(x)| < \epsilon.$$

5. Suppose f is a real-valued continuous function on $[0,1]$ and f is absolutely continuous on $(a,1]$ for every $a \in (0,1)$. Is f necessarily absolutely continuous on $[0,1]$? If f is also of bounded variation on $[0,1]$, is f absolutely continuous on $[0,1]$? If not, give counterexamples.

6. A real-valued function f is *Lipschitz* with constant M if

$$(\forall x, y \in \mathbf{R})\, |f(x) - f(y)| \le M|x - y|.$$

Prove that if f is Lipschitz with constant M if and only if f is absolutely continuous and $|f'| \le M$ a.e.

7. Suppose F_n is a sequence of increasing non-negative right continuous functions on $[0,1]$ such that $\sup_n F_n(1) < \infty$. Let $F \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} F_n$ and suppose $F(1) < \infty$. Prove that

$$F'(x) = \sum_{n=1}^{\infty} F_n'(x)$$

for almost every x.

8. Suppose f is absolutely continuous on $[0,1]$. Write $f(A) \stackrel{\text{def}}{=} \{f(x) : x \in A\}$ for $A \subset [0,1]$. Prove that if A has Lebesgue measure 0, then $f(A)$ has Lebesgue measure 0.

9. If f is real-valued and differentiable at each point of $[0,1]$, is f necessarily absolutely continuous on $[0,1]$? If not, find a counterexample.

10. Find an increasing function f such that $f' = 0$ a.e. but f is not constant on any open interval.

11. Suppose $f : [a,b] \to \mathbf{R}$ is continuous and, for $y \in \mathbf{R}$, let $M(y)$ be the number of points $x \in [a,b]$ such that $f(x) = y$. $M(y)$ may be finite or infinite. Prove that M is Borel measurable and $\int M(y)\, dy$ equals the total variation of f on $[a,b]$.

12. Let $\alpha \in (0,1)$. Find a Borel subset $E \subset [-1,1]$ such that

$$\lim_{r \to 0+} \frac{m(E \cap [-r,r])}{2r} = \alpha.$$

13. Suppose that f is a real-valued continuous function on $[a,b]$ and the derivative D^+f is non-negative on $[a,b]$. Prove that $f(b) \ge f(a)$. Is this true if it is instead assumed that D_+f is non-negative on $[a,b]$?

14. Let

$$f(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{e^{-xy^2}}{1+y^2}\, dy.$$

a. Find the derivative of f.

b. Find an ordinary differential equation that f solves.

c. Find the solution to this ordinary differential equation to determine an explicit value for $f(x)$.

15. Let $(X, \mathcal{A}, \mu)$ be a measure space where $\mu(X) > 0$ and let f be a real-valued integrable function. Define

$$g(x) \stackrel{\text{def}}{=} \int_X |f(y) - x|\, \mu(dx)$$

for $x \in \mathbf{R}$.

a. Prove that g is absolutely continuous.

b. Prove that $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to-\infty} g(x) = \infty$.

c. Find g' and prove that $g(x_0) = \inf_{x\in\mathbf{R}} g(x)$ if and only if

$$\mu(\{y : f(y) > x_0\}) = \mu(\{y : f(y) < x_0\}).$$

16. Suppose $A \subset [0,1]$ has Lebesgue measure zero. Find an increasing function $f : [0,1] \to \mathbf{R}$ that is absolutely continuous, but

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \infty$$

for each $x \in A$.

17. Suppose that μ is a measure on the Borel σ-algebra on$[0,1]$ and every real-valued and continuously differentiable function f satisfies

$$\left| \int f'(x)\, \mu(dx) \right| \leq \left(\int_0^1 f(x)^2\, dx \right)^{1/2}.$$

a. Show that μ is absolutely continuous with respect to Lebesgue measure on $[0,1]$.

b. Let g be the Radon-Nikodym derivative of μ with respect to Lebesgue measure. Prove that there exists a constant $c > 0$ such that

$$(\forall x, y \in [0,1])\ |g(x) - g(y)| \leq c|x-y|^{1/2}.$$

18. Fix $p > 1$ and suppose $f, g \in L^p(\mathbf{R})$. Define

$$H(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |f(x) + tg(x)|^p \, dx$$

for $t \in \mathbf{R}$. Prove that H is a differentiable function and find its derivative.

19. Show that L^∞ is complete.

20. Suppose $1 \leq p \leq \infty$. Prove that the collection of simple functions is dense in L^p.

21. Suppose $p \geq 1$. Prove that

$$\int |f(x)|^p \, dx = \int_0^\infty pt^{p-1} m(\{x : |f(x)| \geq t\}) \, dt.$$

(Hint: see HW 4, exercise 3, which is Exercise 11.3 on textbook p.88.)

22. Suppose f is a measurable function on the measure space $([0,1], \mathcal{B}, m)$, where $\mathcal{B}$ is the Borel σ-algebra and m is Lebesgue measure. Prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

23. When does equality hold in Hölder's inequality?

24. When does equality hold in Minkowski's inequality?

25. Provide a counterexample to the claim that $1 < p < q < \infty$ implies $L^p \subset L^q$.

26. Provide a counterexample to the claim that $1 < p < q < \infty$ implies $L^q \subset L^p$.

27. Define

$$g_n(x) \stackrel{\text{def}}{=} n\chi_{[0,n^{-3}]}(x).$$

a. Show that if $f \in L^2([0,1])$, then

$$\int_0^1 f(x) g_n(x) \, dx \to 0$$

as $n \to \infty$.

b. Show that there exists $f \in L^1([0,1])$ such that $\int_0^1 f(x) g_n(x) \, dx \not\to 0$.

28. Suppose μ is a finite measure on the Borel subsets of $\mathbf{R}$ such that

$$f(x) = \int_{\mathbf{R}} f(x+t) \, \mu(dt), \qquad \text{a.e.},$$

whenever f is real-valued, bounded, and integrable. Prove that $\mu(\{0\}) = 1$.

29. Suppose μ is a measure with $\mu(X) = 1$ and $f \in L^r$ for some $r > 0$, where L^r is defined for $0 < r < 1$ exactly as in equation 15.1 on textbook p.131. Prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left(\int \log |f| \, d\mu \right),$$

where the convention $\exp(-\infty) = 0$ is used if needed.

30. Suppose $1 < p < \infty$ and q is the conjugate exponent to p. Suppose $f_n \to f$ a.e. and $\sup_n \|f_n\|_p < \infty$. Prove that if $g \in L^q$, then

$$\lim_{n \to \infty} \int f_n g = \int f g.$$

31. Does the result in Exercise 30 hold in the case $p = 1$ with $q = \infty$? If not, give a counterexample.

32. Does the result in Exercise 30 hold in the case $p = \infty$ with $q = 1$? If not, give a counterexample.

33. If $f \in L^1(\mathbf{R})$ and $g \in L^p(\mathbf{R})$ for some $p \in [1, \infty)$, prove that

$$\|f * g\|_p \leq \|f\|_1 \, \|g\|_p.$$

34. Suppose $p \in (1, \infty)$ and q is its conjugate exponent. Prove that if $f \in L^p(\mathbf{R})$ and $g \in L^q(\mathbf{R})$, then $f * g$ is uniformly continuous and $f * g(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$.

35. Show that if f and g are continuous with compact support, then $f * g$ is continuous with compact support.

36. Suppose $f \in L^\infty$, $f_h(x) = f(x + h)$, and

$$\lim_{h \to 0} \|f_h - f\|_\infty = 0.$$

Prove that there exists a uniformly continuous function g on $\mathbf{R}$ such that $f = g$ a.e.

37. Let $p \in [0, \infty)$ and suppose μ is a finite measure. Prove that $f \in L^p(\mu)$ if and only if

$$\sum_{n=1}^{\infty} (2^n)^p \, \mu(\{x : |f(x)| > 2^n\}) \; < \; \infty.$$

38. Suppose $\mu(X) = 1$ and f and g are non-negative functions such that $fg \geq 1$ a.e. Prove that

$$\left(\int f\,d\mu\right)\left(\int g\,d\mu\right) \;\geq\; 1.$$

39. Suppose $f : [0,\infty) \to \mathbf{R}$, $f(1) = 0$, f' exists and is continuous and bounded, and $f' \in L^2([1,\infty))$. Let $g(x) \stackrel{\text{def}}{=} f(x)/x$. Prove that $g \in L^2([1,\infty))$.

40. Find an example of a measurable $f : [0,\infty) \to \mathbf{R}$ such that $f(1) = 0$, f' exists and is continuous and bounded, $f' \in L^1([1,\infty))$, but the function $g(x) \stackrel{\text{def}}{=} f(x)/x$ is not in $L^1([1,\infty))$.

41. Prove the *generalized Minkowski inequality*: If $(X, \mathcal{A}, \mu)$ and $(X, \mathcal{B}, \nu)$ are measure spaces, f is measurable with respect to $\mathcal{A} \times \mathcal{B}$, and $1 < p < \infty$, then

$$\left(\int_X \left(\int_Y |f(x,y)|\,\nu(dy)\right)^p \mu(dx)\right)^{1/p} \leq \in_Y \left(\int_X |f(x,y)|^p\,\mu(dx)\right)^{1/p} \nu(dy).$$

This could be rephrased as

$$\left\|\|f\|_{L^1(\nu)}\right\|_{L^p(\mu)} \leq \left\|\|f\|_{L^p(\mu)}\right\|_{L^1(\nu)}.$$

(Put $Y = \{1,2\}$, let $g_1(x) \stackrel{\text{def}}{=} f(x,1)$ and $g_2 \stackrel{\text{def}}{=} f(x,2)$, and take $\nu(dy) = \delta_1(dy) + \delta_2(dy)$ where δ_1, δ_2 are point masses at $1, 2$, respectively, to recover the usual Minkowski inequality, Proposition 15.3 on textbook p.133.)

42. Does the generalized Minkowski inequality in Exercise 41 extend to the case $p = 1$? If not, give a counterexample.

43. Does the generalized Minkowski inequality in Exercise 41 extend to the case $p = \infty$? If not, give a counterexample.

44. Suppose $\alpha \in (0,1)$ and define $K(x) \stackrel{\text{def}}{=} |x|^{-\alpha}$ for $x \in \mathbf{R}$. Note that $K \notin L^p(\mathbf{R})$ for any $p \geq 1$. Prove that if f is non-negative, real-valued, and integrable on $\mathbf{R}$ and

$$g(x) \stackrel{\text{def}}{=} \int_{\mathbf{R}} f(x-t)K(t)\,dt,$$

then g is finite a.e.

45. Suppose $p > 1$ and q is its conjugate exponent, f is an absolutely continuous function on $[0,1]$ with $f' \in L^p$, and $f(0) = 0$. Prove that if $g \in L^q$, then

$$\int_0^1 |fg|\,dx \leq \left(\frac{1}{p}\right)^{1/p} \|f'\|_p \|g\|_q.$$

46. Suppose $f : \mathbf{R} \to \mathbf{R}$ is in L^p for some $p > 1$ and also in L^1. Prove that there exist constants $c > 0$ and $\alpha \in (0, 1)$ such that

$$\int_A |f(x)|\, dx \leq cm(A)^\alpha$$

for every Borel measurable set $A \subset \mathbf{R}$, where m is Lebesgue measure.

47. Suppose $f : \mathbf{R} \to \mathbf{R}$ is integrable and there exist constants $c > 0$ and $\alpha \in (0, 1)$ such that

$$\int_A |f(x)|\, dx \leq cm(A)^\alpha$$

for every Borel measurable set $A \subset \mathbf{R}$, where m is Lebesgue measure. Prove that there exists $p > 1$ such that $f \in L^p$.

48. Suppose $1 < p < \infty$, $f : (0, \infty) \to \mathbf{R}$, and $f \in L^p$ with respect to Lebesgue measure. Define

$$g(x) = \frac{1}{x} \int_0^x f(y)\, dy.$$

Prove that

$$\|g\|_p \leq \frac{p}{p-1} \|f\|_p.$$

(This is known as *Hardy's inequality.*)

49. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and suppose $K : X \times X \to \mathbf{R}$ is measurable with respect to $\mathcal{A} \times \mathcal{A}$. Suppose there exists $M < \infty$ such that

$$\int_X |K(x, y)|\, \mu(dy) \leq M$$

for each x, and

$$\int_X |K(x, y)|\, \mu(dx) \leq M$$

for each y. If f is measurable and real-valued, define

$$Tf(x) \stackrel{\text{def}}{=} \int_X K(x, y) f(y)\, \mu(dy)$$

if the integral exists.

a. Show that $\|Tf\|_1 \leq M\|f\|_1$.

b. If $1 < p < \infty$, show that $\|Tf\|_p \leq M\|f\|_p$.

50. Suppose A and B are two Borel measurable subsets of $\mathbf{R}$, each with finite strictly positive Lebesgue measure. Show that $\chi_A * \chi_B$ is a continuous non-negative function that is not identically equal to zero.

51. Suppose A and B are two Borel measurable subsets of $\mathbf{R}$ with strictly positive Lebesgue measure. Show that

$$C \stackrel{\text{def}}{=} \{x + y : x \in A, y \in B\}$$

contains a non-empty open interval.

52. Suppose $1 < p < \infty$ and q is the conjugate exponent of p. prove that if H is a bounded complex-valued linear functional on L^p, then there exists a complex-valued measurable function $g \in L^q$ such that $H(f) = \int fg$ for all $f \in L^p$ and $\|H\| = \|g\|_q$.