Math 5051
Measure Theory and Functional Analysis I
Homework Assignment 4

Prof. Wickerhauser

Due **Monday, November 2nd, 2015**

Please do Exercises 3*, 6, 8, 9, 12*, 13, 16*, 22, 24, 29*, 31, 37.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. These exercises originate from that source.

1. State and prove a version of the Fubini theorem for complex-valued functions.

2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let $f \geq 0$ be measurable with respect to $\mathcal{A} \times \mathcal{B}$. Let $g(x) = \sup_{y \in Y} f(x, y)$ and suppose $g(x) < \infty$ for each x. Is g necessarily measurable with respect to $\mathcal{A}$? If not, find a counterexample.

3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable. Prove the equality

$$\int_{\mathbf{R}} |f(x)|\, dx = \int_0^\infty m(\{x : |f(x)| \geq t\})\, dt,$$

where m is Lebesgue measure.

4. Let A be a Lebesgue measurable subset of $[0, 1] \times [0, 1]$ with $m_2(A) = 1$, where m_2 is two-dimensional Lebesgue measure. Show that for almost every $x \in [0, 1]$ (with respect to one-dimensional Lebesgue measure) the set $s_x(A)$ has one-dimensional Lebesgue measure 1.

5. Let $f : [0, 1] \times [0, 1] \to \mathbf{R}$ be such that for every $x \in [0, 1]$ the function $y \mapsto f(x, y)$ is Lebesgue measurable on $[0, 1]$ and for every $y \in [0, 1]$ the function $x \mapsto f(x, y)$ is continuous on $[0, 1]$. Prove that f is measurable with respect to the completion of the product σ-algebra $\mathcal{L} \times \mathcal{L}$ on $[0, 1] \times [0, 1]$. Here $\mathcal{L}$ is the Lebesgue σ-algebra on $[0, 1]$.

6. Suppose f is real-valued and integrable with respect to two-dimensional Lebesgue measure on $[0,1] \times [0,1]$ and

$$\int_0^a \int_0^b f(x,y)\, dy\, dx = 0$$

for all $a \in [0,1]$ and $b \in [0,1]$. Prove that $f = 0$ a.e.

7. Let

$$f(x,y) \stackrel{\text{def}}{=} \frac{x^2 - y^2}{(x^2+y^2)^{3/4}} \log(4 + \sin x)$$

for $x, y \in [0,1]$. Prove that

$$\int_0^1 \int_0^1 f(x,y)\, dy\, dx = \int_0^1 \int_0^1 f(x,y)\, dx\, dy.$$

8. Let $X = Y = [0,1]$ and let $\mathcal{B}$ be the Borel σ-algebra. Let m be Lebesgue measure and μ counting measure on $[0,1]$.

   a. If $D = \{(x,y) : x = y\}$, show that D is measurable with respect to $\mathcal{B} \times \mathcal{B}$.

   b. Show that

$$\int_X \int_Y \chi_D(x,y)\, \mu(dy)\, m(dx) \neq \int_X \int_Y \chi_D(x,y)\, m(dx)\, \mu(dy).$$

   Why does this not contradict the Fubini theorem?

9. Let $X = Y = \mathbf{R}$ and let $\mathcal{B}$ be the Borel σ-algebra. Define

$$f(x,y) = \begin{cases} 1, & x \geq 0 \text{ and } x \leq y < x+1; \\ -1, & x \geq 0 \text{ and } x+1 \leq y < x+2; \\ 0, & \text{otherwise.} \end{cases}$$

   Show that

$$\int \int f(x,y)\, dx\, dy \neq \int \int f(x,y)\, dy\, dx.$$

   Why does this not contradict the Fubini theorem?

10. Find a real-valued function f that is integrable on $[0,1]^2$ such that

$$\int_0^a \int_0^1 f(x,y)\, dy\, dx = 0, \qquad \int_0^1 \int_0^b f(x,y)\, dx\, dy = 0$$

for every $a, b \in [0,1]$, but f is not zero almost everywhere with respect to to-dimensional Lebesgue measure.

11. Let μ be a finite measure on $\mathbf{R}$ and let $f(x) = \mu((-\infty, x])$. Show that

$$\int [f(x+c) - f(x)]\, dx = c\mu(\mathbf{R}).$$

12. Use

$$\frac{1}{x} = \int_0^\infty e^{-xy}\, dy$$

and the Fubini theorem to calculate

$$\int_0^b \int_0^\infty e^{-xy} \sin x\, dy\, dx$$

in two different ways. Use that to prove

$$\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(Hint: recall that

$$\int e^{au} \sin u\, du = \frac{e^{au}(a \sin u - \cos u)}{1 + a^2} + C.)$$

13. Let $X = \{1, 2, \ldots\}$ and let μ be counting measure on X. Define $f : X \times X \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} 1, & \text{if } x = y; \\ -1, & \text{if } x = y + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Show that

$$\int_X \int_X f(x,y)\, \mu(dx)\, \mu(dy) \neq \int_X \int_X f(x,y)\, \mu(dy)\, \mu(dx).$$

Why does this not contradict the Fubini theorem?

14. Let $\{a_n\}$ and $\{r_n\}$ be two sequences of real numbers such that $\sum_{n=1}^\infty |a_n| < \infty$. Prove that

$$\sum_{n=1}^\infty \frac{a_n}{\sqrt{|x - r_n|}}$$

converges absolutely for almost every $x \in \mathbf{R}$.

15. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Prove that if λ is a measure on $\mathcal{A} \times \mathcal{B}$ such that $\lambda(A \times B) = \mu(A)\nu(B)$ whenever $A \in \mathcal{A}$ an $B \in \mathcal{B}$, then $\lambda = \mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$.

16. Let $S = \{e^{i\theta} : 0 \le \theta < 2\pi\}$ be the unit circle and define a measure μ on S by

$$\mu(A) \stackrel{\text{def}}{=} m(\{\theta \in [0, 2\pi) : e^{i\theta} \in A\}),$$

where m is Lebesgue measure on $[0, 2\pi)$. Let m_2 be two-dimensional Lebesgue measure. Show that if A is a Borel subset of S and $R > 0$, then

$$m_2(\{re^{i\theta} : 0 < r < R,\ e^{i\theta} \in A\}) \ = \ \mu(A)R^2/2.$$

17. Use Exercise 16 to prove that if f is a continuous real-valued function with support in the ball $B(0, R) = \{(x, y) : x^2 + y^2 < R^2\}$, then

$$\iint_{B(0,R)} f(x, y)\, dy\, dx = \int_0^{2\pi} \int_0^R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

18. Prove that

$$I \stackrel{\text{def}}{=} \int_0^\infty e^{-x^2/2}\, dx = \sqrt{\pi/2}$$

by using Exercise 17 with $f(x, y) = e^{-(x^2+y^2)/2}\chi_{B(0,R)}(x, y)$, then letting $R \to \infty$. Justify the equality

$$I^2 = \lim_{R\to\infty} \iint_{B(0,R)} f(x, y)\, dy\, dx$$

as part of your proof.

19. Let $M = (M_{ij})_{i,j=1}^n$ be an $n \times n$ matrix and $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ a vector. If A is a Borel subset of $\mathbf{R}^n$, define $M(A) \stackrel{\text{def}}{=} \{Mx : x \in A\}$ where Mx is the usual matrix-vector product. Let m_n be n-dimensional Lebesgue measure on R^n.

a. If $c \in \mathbf{R}$ and

$$M_{ij} \stackrel{\text{def}}{=} \begin{cases} c, & \text{if } i = j = 1; \\ 1, & \text{if } i = j > 1; \\ 0, & \text{if } i \ne j, \end{cases}$$

show that $m_n(M(A)) = |c|\, m_n(A) = |\det M|\, m_n(A)$. (Multiplication by M multiplies the first coordinate by c.)

b. If $1 \le k \le n$ and

$$M_{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = 1 \text{ and } j = k; \\ 1, & \text{if } j = 1 \text{ and } i = k; \\ 1, & \text{if } i = j \text{ and neither equals } k; \\ 0, & \text{otherwise,} \end{cases}$$

show that $m_n(M(A)) = m_n(A) = |\det M|\, m_n(A)$. (Multiplication by M interchanges the first and kth coordinates.)

c. If $c \in \mathbf{R}$ and

$$M_{ij} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } i = j; \\ c, & \text{if } i = 1 \text{ and } j = 2; \\ 0, & \text{otherwise,} \end{cases}$$

show that $m_n(M(A)) = m_n(A) = |\det M|\, m_n(A)$. (Multiplication by M replaces x_1 by $x_1 + cx_2$.)

d. Since every $n \times n$ matrix can be written as the product of matrices of the form in (a), (b), or (c), conclude that if M is any $n \times n$ matrix, then $m_n(M(A)) = |\det M|\, m_n(A)$.

e. If M is an orthogonal matrix, namely $MM^T = M^T M = Id$, show that $m_n(M(A)) = m_n(A)$. (Multiplication by an orthogonal matrix is a rotation of $\mathbf{R}^n$.)

20. Suppose μ is a signed measure. Prove that A is a null set with respect to μ if and only if $|\mu|(A) = 0$.

21. Let μ be a signed measure. Define

$$\int f\, d\mu \stackrel{\text{def}}{=} \int f\, d\mu^+ - \int f\, d\mu^-.$$

Prove that

$$\left| \int f\, d\mu \right| \leq \int |f|\, d|\mu|.$$

22. Let μ be a signed measure on $(X, \mathcal{A})$. Prove that

$$|\mu|(A) = \sup \left\{ \left| \int_A f\, d\mu \right| : |f| \leq 1,\ f \text{ is } \mathcal{A}\text{-measurable} \right\},$$

for any $A \in \mathcal{A}$.

23. Let $(X, \mathcal{A})$ be a measurable space. Suppose $\lambda = \mu - \nu$, where μ and ν are finite positive measures. Prove that $\mu(A) \geq \lambda^+(A)$ and $\nu(A) \geq \lambda^-(A)$ for every $A \in \mathcal{A}$.

24. Let $(X, \mathcal{A})$ be a measurable space. Prove that if μ and ν are finite **signed** measures, then $|\mu + \nu|(A) \leq |\mu|(A) + |\nu|(A)$ for every $A \in \mathcal{A}$.

25. Suppose that μ is a signed measure on $(X, \mathcal{A})$. Prove that if $A \in \mathcal{A}$, then

$$\mu^+(A) = \sup\{\mu(B) : B \in \mathcal{A},\ B \subset A\}$$

and

$$\mu^-(A) = -\inf\{\mu(B) : B \in \mathcal{A},\ B \subset A\}.$$

26. Suppose that μ is a signed measure on $(X, \mathcal{A})$. Prove that if $A \in \mathcal{A}$, then

$$|\mu|(A) = \sup \left\{ \sum_{j=1}^{n} |\mu(B_j)| : \text{each } B_j \in \mathcal{A}, \text{ the } B_j \text{ are disjoint, } \bigcup_{j=1}^{n} B_j = A \right\}.$$

27. Generalize the Radon-Nikodym theorem to signed measures: Let $(X, \mathcal{A})$ be a measurable space, ν a finite signed measure, and μ a finite positive measure. Suppose that $\mu(A) = 0$ implies $\nu(A) = 0$ for $A \in \mathcal{A}$. Show that there exists an integrable real-valued function f such that $\nu(A) = \int_A f \, d\mu$ for all $A \in \mathcal{A}$.

28. State and prove a generalization of the Lebesgue decomposition theorem for signed measures.

29. Define a *complex measure* μ on a measurable space $(X, \mathcal{A})$ to be a bounded map from $\mathcal{A}$ to $\mathbf{C}$ such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n)$$

whenever $A_n \in \mathcal{A}$ for all n and the A_n are pairwise disjoint.

State and prove a Radon-Nikodym theorem for complex measures.

30. Let μ be a complex measure on a measurable space $(X, \mathcal{A})$ (see Exercise 29). The *total variation measure* is defined to be a positive measure $|\mu|$ such that $d|\mu| = |f| \, d\rho$, where ρ is a positive measure and f is a measurable function such that $d\mu = f \, d\rho$.

The quantity $|\mu|(X)$ is called the *total variation of* μ.

Prove that the definition of $|\mu|$ is independent of the choice of ρ, that is, if $d\mu = f_1 \, d\rho_1 = f_2 \, d\rho_2$, then $|f_1| \, d\rho_1 = |f_2| \, d\rho_2$.

31. Let $(X, \mathcal{A})$ be a measurable space and let μ and ν be finite measures. Say that μ and ν are *equivalent measures* if $\mu \ll \nu$ and $\nu \ll \mu$.

Show that μ and ν are equivalent measures if and only if there exists a μ-integrable function f that is strictly positive μ-a.e. and such that $d\nu = f \, d\mu$.

32. Suppose μ and ν are two finite positive measures such that ν is absolutely continuous with respect to μ. Let $\rho = \mu + \nu$. Note that $\mu(A) \leq \rho(A)$ and $\nu(A) \leq \rho(A)$ for each measurable A. In particular, $\mu \ll \rho$ and $\nu \ll \rho$.

Prove that if $f = d\mu/d\rho$ and $g = d\nu/d\rho$, then f is strictly positive μ-a.e., $f + g = 1$, and $d\nu = (g/f) \, d\mu$.

33. If μ is a signed measure on $(X, \mathcal{A})$ and $|\mu|$ is the total variation measure, prove that there exists a real-valued $\mathcal{A}$-measurable function f such that $|f| = 1$ μ-a.e. and $d\mu = f\, d|\mu|$.

34. Suppose μ, ν, and ρ are signed measures, $\nu \ll \mu$, and $\rho \ll \nu$. Prove that $\rho \ll \mu$ and

$$\frac{d\rho}{d\mu} = \frac{d\rho}{d\nu} \cdot \frac{d\nu}{d\mu}.$$

35. Let μ be a positive measure and ν a signed measure. Prove that $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

36. Suppose λ_n is a sequence of finite positive measures on a measurable space $(X, \mathcal{A})$ with $\sup_n \lambda_n(X) < \infty$, and μ is another finite positive measure on $(X, \mathcal{A})$. Suppose $\lambda_n = f_n\, d\mu + \nu_n$ is the Lebesgue decomposition of λ_n for each n; in particular, $\nu_n \perp \mu$.

    If $\lambda \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \lambda_n$ is a finite measure, show that

$$\lambda = \left(\sum_{n=1}^{\infty} f_n \right) d\mu + \sum_{n=1}^{\infty} \nu_n$$

    is the Lebesgue decomposition of λ.

37. Suppose X is a set and $\mathcal{E} \subset \mathcal{F}$ are two σ-algebras of subsets of X. Let μ, ν be two finite positive measures on $(X, \mathcal{F})$ and suppose $\nu \ll \mu$. Let $\bar{\mu}$ and $\bar{\nu}$ be the restrictions of μ and ν, respectively, to $\mathcal{E}$.

    Find an example within the above framework for which $d\bar{\nu}/d\bar{\mu} \neq d\nu/d\mu$. This will demonstrate that the Radon-Nikodym derivative may depend on the σ-algebra.

    (Hint: exploit the requirement that the Radon-Nikodym derivative must be a measurable function. Find an example $\mathcal{E} \subset \mathcal{F}$ where only constant functions are $\mathcal{E}$-measurable, but there are non-constant $\mathcal{F}$-measurable functions.)

38. Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose that $\mathcal{E} \subset \mathcal{F}$ is a sub-σ-algebra, namely a subset that is also a σ-algebra. Suppose that f is a non-negative integrable function that is measurable with respect to $\mathcal{F}$. Define

$$\nu(A) \stackrel{\text{def}}{=} \int_A f\, d\mu, \qquad A \in \mathcal{E},$$

    and let $\bar{\mu}$ be the restriction of μ to $\mathcal{E}$.

    a. Prove that $\nu \ll \bar{\mu}$.

b. Since ν and $\bar{\mu}$ are measures on $\mathcal{E}$, then $g \stackrel{\text{def}}{=} d\nu/d\bar{\mu}$ is measurable with respect to $\mathcal{E}$. Prove that

$$\int_A g \, d\mu = \int_A f \, d\mu$$

whenever $A \in \mathcal{E}$.

(This g is called the *conditional expectation* of f with respect to $\mathcal{E}$ and is denoted by $g = \mathbf{E}[\, f \,|\, \mathcal{E}\,]$. If f is integrable and real-valued but not necessarily non-negative, then

$$\mathbf{E}[\, f \,|\, \mathcal{E}\,] \stackrel{\text{def}}{=} \mathbf{E}[\, f^+ \,|\, \mathcal{E}\,] - \mathbf{E}[\, f^- \,|\, \mathcal{E}\,].)$$

c. Show that $f = g$ in part (b) if and only if f is $\mathcal{E}$-measurable.

d. Prove that if h is $\mathcal{E}$-measurable and $\int_A h \, d\mu = \int_A f \, d\mu$ for all $A \in \mathcal{E}$, then $h = g$ μ-a.e., where g is the conditional expectation defined in part (b).