Math 5051
Measure Theory and Functional Analysis I
Homework Assignment 3

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Due Monday, October 12th, 2015

Please do Exercises 3*, 4, 5, 6, 8*, 11*, 17, 20, 21, 22, 27*.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. These exercises originate from that source.

1. Prove these basic change of variable formulas for Lebesgue integrals: If $f : \mathbf{R} \to \mathbf{R}$ is integrable and $a \neq 0$, then

$$\int_{\mathbf{R}} f(x+a)\,dx = \int_{\mathbf{R}} f(x)\,dx \quad \text{and} \quad \int_{\mathbf{R}} f(ax)\,dx = |a|^{-1} \int_{\mathbf{R}} f(x)\,dx.$$

2. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Suppose f is non-negative and integrable. Prove that if $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$ and

$$\epsilon + \int_A f\,d\mu > \int f\,d\mu.$$

3. Suppose A is a Borel measurable subset of $[0,1]$, m is Lebesgue measure, and $\epsilon \in (0,1)$. Prove that there exists a continuous function $f : [0,1] \to \mathbf{R}$ such that $0 \leq f \leq 1$ and

$$m(\{x : f(x) \neq \chi_A(x)\}) < \epsilon$$

4. Suppose f is a non-negative integrable function on a measure space $(X, \mathcal{A}, \mu)$. Prove that

$$\lim_{t\to\infty} t\mu(\{x : f(x) \geq t\}) = 0.$$

5. Let m be Lebesgue measure. Find a non-negative measurable function f on $[0, 1]$ such that

$$\lim_{t\to\infty} t\, m(\{x : f(x) \geq t\}) = 0$$

but f is not integrable.

6. Suppose μ is a finite measure. Prove that a measurable non-negative function f is integrable if and only if

$$\sum_{n=1}^{\infty} \mu(\{x : f(x) \geq n\}) < \infty.$$

7. Let μ be a measure, not necessarily σ-finite, and suppose f is real-valued and integrable with respect to μ. Prove that $A = \{x : f(x) \neq 0\}$ has *σ-finite measure*, namely that there exists $F_n \uparrow A$ such that $\mu(F_n) < \infty$ for each n.

8. Recall that a function $f : \mathbf{R} \to \mathbf{R}$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

whenever $x, y \in \mathbf{R}$ and $\lambda \in [0, 1]$.

   a. Prove that if f is convex and $x \in \mathbf{R}$, there exists $c \in \mathbf{R}$ such that $f(y) \geq f(x) + c(y - x)$ for all $y \in \mathbf{R}$. (So the graph of f lies above the line with slope c passing through $(x, f(x))$.)

   b. Let $(X, \mathcal{A}, \mu)$ be a measure space, suppose $\mu(X) = 1$, and let $f : \mathbf{R} \to \mathbf{R}$ be convex. Let $g : X \to \mathbf{R}$ be integrable. Prove that

$$f\left(\int g\, d\mu\right) \leq \int_X f \circ g\, d\mu.$$

   (This called *Jensen's inequality.*)

9. Suppose $f : \mathbf{R} \to \mathbf{R}$ satifies

$$f\left(\int_0^1 g(x)\, dx\right) \leq \int_0^1 f(g(x))\, dx$$

whenever g is bounded and measurable. Prove that f is convex (see Exercise 8).

10. Suppose $g : [0,1] \to \mathbf{R}$ is bounded and measurable and

$$\int_0^1 f(x)g(x)\,dx = 0$$

whenever f is continuous and $\int_0^1 f(x)\,dx = 0$. Prove that there is some constant c such that $g = c$ a.e.

11. Find a measurable function $f : [0,1] \to \mathbf{R}$ such that $\overline{R}(f) \neq \int_0^1 f(x)\,dx$ and $\underline{R}(f) \neq \int_0^1 f(x)\,dx$.

12. Find a function $f : (0,1] \to \mathbf{R}$ that is continuous, is not Lebesgue integrable, but for which the improper Riemann integral

$$\lim_{a \to 0+} R(f\chi_{(a,1]})$$

exists and is finite.

13. Suppose $f : [0,1] \to \mathbf{R}$ that is Lebesgue integrable, f is bounded on $(a,1]$ for each $a \in (0,1)$, and the improper Riemann integral

$$\lim_{a \to 0+} R(f\chi_{(a,1]})$$

exists. Show that the limit is equal to (the Lebesgue integral) $\int_0^1 f(x)\,dx$.

14. Divide $[a,b]$ into 2^n equal subintervals and pick a point x_i out of each subinterval. Let μ_n be the measure defined by

$$\mu_n(A) = 2^{-n} \sum_{i=1}^{2^n} \delta_{x_i}(A),$$

where δ_y is the point mass at y. Then

$$\int_a^b f(x)\,\mu(dx) = \sum_{i=1}^{2^n} f(x_i)2^{-n}$$

for any bounded measurable $f : [a,b] \to \mathbf{R}$, so this measure gives the Riemann sum approximation to $R(f)$.

a. Prove that $\mu_n([0,x]) \to m([0,x])$ for every $x \in [0,1]$. Conclude by Exercise 7.24 on textbook p.60 (which is Exercise 39 of HW 2) that

$$\int f\,d\mu_n \to \int_0^1 f\,dx \tag{1}$$

whenever f is continuous.

b. Use Exercise 7.25 on textbook p.60 (which is Exercise 39 of HW 2) to prove that if $f : [a, b] \to \mathbf{R}$ is a bounded and measurable function whose set of discontinuities has measure 0, then the Riemann sum approximation given in Equation 1 converges to the Lebesgue integral of f. This provides an alternative proof of Step 2 of Theorem 9.1 on textbook p.71.

15. Let f be a bounded, real-valued, and measurable function. Prove that if

$$\overline{f} = \lim_{\delta \to 0} \sup_{|y-x|<\delta, a \le y \le b} f(y),$$

then $\overline{f} = T$ a.e., using the notation of Theorem 9.1 on textbook p.70. Conclude that $\overline{f}$ is Lebesgue measurable.

16. Define

$$\underline{f} = \lim_{\delta \to 0} \inf_{|y-x|<\delta, a \le y \le b} f(y),$$

and let $\overline{f}$ be as in Exercise 15.

a. Suppose that the set of discontinuities of a bounded real-valued measurable function f has positive Lebesgue measure. Prove that there exists $\epsilon > 0$ such that if

$$A_\epsilon \stackrel{\text{def}}{=} \{x \in [a, b] : \overline{f}(x) - \underline{f}(x) > \epsilon\},$$

then $m(A_\epsilon) > 0$.

b. Prove that $U(P, f) - L(P, f) > \epsilon m(A_\epsilon)$ for every partition P on $[a, b]$, using the notation of Theorem 9.1 of textbook p.70. Conclude that f is not Riemann integrable. This provides an alternative proof of Step 1 of Theorem 9.1.

17. A real-valued function on a metric space is *lower semicontinuous* if $\{x : f(x) > a\}$ is open for all $a \in \mathbf{R}$, and is *upper semicontinuous* if $\{x : f(x) < a\}$ is open for all $a \in \mathbf{R}$.

a. Prove that if f_n is a sequence of real-valued continuous functions increasing to f, then f is lower semicontinuous.

b. Find a bounded lower semicontinuous function $f : [0, 1] \to \mathbf{R}$ such that f is continuous everywhere except $x = 1/2$.

c. Find a bounded lower semicontinuous function $f : [0, 1] \to \mathbf{R}$ such that the set of discontinuities of f is $\mathbf{Q} \cap [0, 1]$, the rationals in $[0, 1]$.

d. Find a bounded lower semicontinuous function $f : [0, 1] \to \mathbf{R}$ such that the set of discontinuities of f has positive measure.

e. Does there exist a bounded lower semicontinuous function $f : [0,1] \to \mathbf{R}$ such that f is discontinuous a.e.?

18. Find a sequence $f_n : [0,1] \to [0,1]$ of continuous functions such that $f_n \uparrow f$ but f is not Riemann integrable. (This shows that there is no monotone convergence theorem or dominated convergence theorem for the Riemann integral.)

19. Let $M > 0$ and let $\mathcal{B}$ be the σ-algebra on $X = [-M, M] \times [-M, M] \subset \mathbf{R}^2$ generated by the collection of closed rectangles

$$\{[a,b] \times [c,d] : -M \le a \le b \le M, -M \le c \le d \le M\}.$$

Suppose μ is a measure on $(X, \mathcal{B})$ such that

$$\mu([a,b] \times [c,d]) = (b-a)(d-c).$$

Prove that if f is continuous on $\mathbf{R}^2$ with support in X, then the Lebesgue integral of f with respect to μ is equal to the double Riemann integral of f and the two iterated Riemann integrals of f.

20. Let $(X, \mathcal{A}, \mu)$ be a measure space. Say that a sequence f_n of measurable real-valued functions on X is *Cauchy in measure* if, for all $\epsilon > 0$ and $a > 0$, there exists N such that

$$m, n \ge N \;\Rightarrow\; \mu(\{x : |f_n(x) - f_m(x)| > a\}) < \epsilon.$$

Prove that if f_n is Cauchy in measure, then it converges in measure.

21. Suppose $\mu(X) < \infty$. Define

$$d(f,g) \stackrel{\text{def}}{=} \int \frac{|f-g|}{1+|f-g|}\, d\mu.$$

a. Prove that d is a metric on the space of measurable functions, except that $d(f,g) = 0$ only implies that $f = g$ a.e., not necessarily everywhere.

b. Prove that $f_n \to f$ in measure if and only $d(f_n, f) \to 0$.

22. Prove that if $f_n \to f$ in measure and each f_n is non-negative, then

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

23. Prove that if A_n is measurable with $\mu(A_n) < \infty$ for each n, and $\chi_{A_n} \to f$ in measure, then there exists a measurable set A such that $f = \chi_A$ a.e.

24. Suppose for each $\epsilon > 0$ there exists a measurable set F such that $\mu(F^c) < \epsilon$ and f_n converges to f uniformly on F. Prove that $f_n \to f$ a.e.

25. Suppose that f_n and f are measurable functions such that for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \infty.$$

Prove that $f_n \to f$ a.e.

26. Let f_n be a sequence of measurable functions and define

$$g_n(x) \stackrel{\text{def}}{=} \sup_{m \geq n} |f_m(x) - f_n(x)|.$$

Prove that if $g_n \to 0$ in measure, then f_n converges a.e.

27. If $(X, \mathcal{A}, \mu)$ is a measure space and f_n is a sequence of real-valued measurable functions such that $\int f_n g \, d\mu \to 0$ for every integrable g, is it necessarily true that $f_n \to 0$ in measure? Give a proof or find a counterexample.

28. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and X is a countable set. Let $\{f_n\}$ be a sequence of measurable functions. Prove that if $f_n \to f$ in measure, then $f_n \to f$ a.e.