Math 5051
Measure Theory and Functional Analysis I
Homework Assignment 2

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Due Friday, September 25th, 2015

Please do Exercises 1, 4*, 7, 9*, 11, 12, 13, 16, 21*, 26, 28, 31, 32, 33, 36, 37.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. These exercises originate from that source.

1. Suppose $(X, \mathcal{A})$ is a measurable space, $f : X \to \mathbf{R}$ is a function, and $\{x : f(x) > r\} \in \mathcal{A}$ for all $r \in \mathbf{Q}$. Prove that f is measurable.

2. Let $f : (0, 1) \to \mathbf{R}$ be such that for every $x \in (0, 1)$ there exist $r > 0$ and a Borel measurable function g, both depending on x, such that f and g agree on $(x - r, x + r) \cap (0, 1)$. Prove that f is Borel measurable.

3. Suppose f_n is a sequence of measurable functions. Prove that

$$A = \{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

   is a measurable set.

4. If $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable, prove that there exists a Borel measurable function g such that $f = g$ a.e.

5. Give an example of a collection of measurable non-negative functions $\{f_\alpha\}_{\alpha\in A}$ such that if g is defined by $g(x) = \sup_{\alpha\in A} f_\alpha(x)$, then g is finite for all x but g is non-measurable. (Hint: A is allowed to be uncountable.)

6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable and $g : \mathbf{R} \to \mathbf{R}$ is continuous. Prove that $g \circ f$ is Lebesgue measurable. Is this true if g is Borel measurable instead of continuous? Is this true if g is Lebesgue measurable instead of continuous?

7. Suppose $f : \mathbf{R} \to \mathbf{R}$ is Borel measurable. Define $\mathcal{A}$ to be the smallest σ-algebra containing the sets $\{x : f(x) > a\}$ for every $a \in \mathbf{R}$. Suppose g is measurable with respect to $\mathcal{A}$, namely that $(\forall a \in \mathbf{R})\{x : g(x) > a\} \in \mathcal{A}$. Prove that there exists a Borel measurable function $h : \mathbf{R} \to \mathbf{R}$ such that $g = h \circ f$.

8. It is known that there exist discontinuous real-valued functions f such that

$$(\forall x, y \in \mathbf{R})\ f(x + y) = f(x) + f(y). \tag{1}$$

   (An example may be constructed using Zorn's lemma.) Prove that if f satisfies (1) and in addition f is Lebesgue measurable, then f is continuous.

9. Verify Equation (6.5) on textbook p.48. Namely, for measure space $(X, \mathcal{A}, \mu)$, show that if $\sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{j=1}^{m} b_j \chi_{B_j}$ for $A_i, B_j \in \mathcal{A}$ and $a_i, b_j \in \mathbf{R}$, then

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{j=1}^{m} b_j \mu(B_j).$$

10. Suppose f is non-negative and measurable and μ is σ-finite. Show there exist simple functions s_n increasing to f at each point such that $\mu(\{x : s_n(x) \neq 0\}) < \infty$ for each n.

11. Let f be a non-negative measurable function. Prove that

$$\lim_{n \to \infty} \int \min(f, n) = \int f.$$

12. Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose μ is σ-finite. Suppose f is integrable. Prove that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |f(x)|\, \mu(dx) < \epsilon$$

   whenever $\mu(A) < \delta$.

13. Suppose $\mu(X) < \infty$ and f_n is a sequence of bounded real-valued measurable functions that converge to f uniformly. Prove that

$$\int f_n \, d\mu \to \int f \, d\mu.$$

This is sometimes called the *bounded convergence theorem.* (Hint: prove this with or without the dominated convergence theorem.)

14. If f_n is a sequence of non-negative integrable functions such that $f_n(x)$ decreases to $f(x)$ for every x, prove that $\int f_n \, d\mu \to \int f \, d\mu$.

15. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and suppose f is a non-negative, measurable function that is finite at each point of X, but is not necessarily integrable. Prove that there exists a continuous increasing function $g : [0, \infty) \to [0, \infty)$ such that $\lim_{x\to\infty} g(x) = \infty$ and $g \circ f$ is integrable.

16. State and prove a version of the Dominated Convergence Theorem for complex-valued functions. (Hint: prove this as a corollary to the dominated convergence theorem for real-valued functions.)

17. Suppose f_n, g_n, f, g are integrable, $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$ for each n, and $\int g_n \to \int g$. Prove that $\int f_n \to \int f$. (Hint: use the dominated convergence theorem.)

18. Give an example of a sequence of non-negative functions f_n tending to 0 pointwise such that $\int f_n \to 0$ but there is no integrable function g such that $f_n \leq g$ for all n.

19. Suppose $(X, \mathcal{A}, \mu)$ is a measure space, f and each f_n is integrable and non-negative, $f_n \to f$ a.e, and $\int f_n \to \int f$. Prove that for each $A \in \mathcal{A}$,
$$\int_A f_n \, d\mu \to \int_A f \, d\mu.$$

20. Suppose f_n and f are integrable, $f_n \to f$ a.e., and $\int |f_n| \to \int |f|$. Prove that
$$\int |f_n - f| \to 0.$$

21. Suppose $f : \mathbf{R} \to \mathbf{R}$ is integrable, $a \in \mathbf{R}$, and we define
$$F(x) \stackrel{\text{def}}{=} \int_a^x f(y) \, dy.$$
Show that F is a continuous function.

22. Let f_n be a sequence of non-negative Lebesgue measurable functions on $\mathbf{R}$. Is it necessarily true that
$$\limsup_{n\to\infty} \int f_n \, dx \leq \int \limsup_{n\to\infty} f_n \, dx?$$
If not, give a counterexample.

23. Find the limit
$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^{-n} \log(2+\cos(x/n))\,dx$$
and justify your reasoning.

24. Find the limit
$$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^{n} \log(2+\cos(x/n))\,dx$$
and justify your reasoning.

25. Prove that the limit exists and find its value:
$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} \log(2+\cos(x/n))\,dx$$

26. Prove that the limit exists and determine its value:
$$\lim_{n\to\infty} \int_0^\infty ne^{-nx}\sin(1/x)\,dx$$

27. Let $g:\mathbf{R}\to\mathbf{R}$ be integrable and let $f:\mathbf{R}\to\mathbf{R}$ be bounded, measurable, and continuous at 1. Prove that
$$\lim_{n\to\infty} \int_{-n}^n f\left(1+\frac{x}{n^2}\right) g(x)\,dx$$
exists and determine its value.

28. Suppose $\mu(X)<\infty$, f_n converges to f uniformly, and each f_n is integrable. Prove that f is integrable and $\int f_n \to \int f$. Is the condition $\mu(X)<\infty$ necessary?

29. Prove that
$$\sum_{k=1}^\infty \frac{1}{(p+k)^2} = -\int_0^1 \frac{x^p}{1-x}\log x\,dx$$
for $p>0$. (Hint: use the fundamental theorem of calculus, to be proved in textbook Chapter 8, namely if f is continuous on $[a,b]$ and F is differentiable on $[a,b]$ with derivative f, then $\int_a^b f(x)\,dx = F(b)-F(a)$.)

30. Let f_n be a sequence of measurable real-valued functions on $[0,1]$ that is uniformly bounded.

    a. Show that if A is a Borel subset of $[0,1]$, then there exists a subsequence n_j such that $\int_A f_{n_j}(x)\,dx$ converges.

b. Show that if $\{A_i\}$ is a countable collection of Borel subsets of $[0,1]$, then there exists a subsequence n_j such that $\int_{A_i} f_{n_j}(x)\,dx$ converges for each i.

c. Show that there exists a subsequence n_j such that $\int_A f_{n_j}(x)\,dx$ converges for each Borel subset A of $[0,1]$.

31. Let $(X, \mathcal{A}, \mu)$ be a measure space. A sequence of measurable functions $\{f_n\}$ is *uniformly integrable* if, given $\epsilon > 0$, there exists M such that

$$\int_{\{x:|f_n(x)|>M\}} |f_n(x)|\,d\mu \;<\; \epsilon$$

for each n. The sequence is *uniformly absolutely continuous* if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\int_A f_n\,d\mu\right| \;<\; \epsilon$$

for each n and each $A \in \mathcal{A}$ with $\mu(A) < \delta$.

Suppose that μ is a finite measure. Prove that $\{f_n\}$ is uniformly integrable if and only if $\sup_n \int |f_n|\,d\mu < \infty$ and $\{f_n\}$ is uniformly absolutely continuous.

32. Suppose μ is a finite measure, $f_n \to f$ a.e., and $\{f_n\}$ is uniformly integrable (see Exercise 31). Prove that $\int |f_n - f| \to 0$. (This is known as the *Vitali convergence theorem.*)

33. Suppose μ is a finite measure, $f_n \to f$ a.e., each f_n is integrable, and $\int |f_n - f| \to 0$. Prove that $\{f_n\}$ is uniformly integrable (see Exercise 31).

34. Suppose μ is a finite measure and for some $\epsilon > 0$,

$$\sup_n \int |f_n|^{1+\epsilon}\,d\mu \;<\; \infty.$$

Prove that $\{f_n\}$ is uniformly integrable (see Exercise 31).

35. Suppose $\{f_n\}$ is a uniformly integrable sequence of functions defined on $[0,1]$. Prove that there is a subsequence n_j such that $\int_0^1 f_{n_j} g\,dx$ converges whenever g is a real-valued bounded measurable function.

36. Suppose μ_n is a sequence of measures on $(X, \mathcal{A})$ such that $\mu_n(X) = 1$ for all n and $\mu_n(A)$ converges as $n \to \infty$ for each $A \in \mathcal{A}$. Call the limit $\mu(A)$.

a. Prove that μ is a measure.

b. Prove that $\int f\,d\mu_n \to \int f\,d\mu$ whenever f is bounded and measurable.

c. Prove that

$$\int f\,d\mu \leq \liminf_{n\to\infty} \int f\,d\mu_n$$

whenever f is non-neagative and measurable.

37. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be non-negative and integrable. Define ν on $\mathcal{A}$ by

$$\nu(A) \stackrel{\text{def}}{=} \int_A f\,d\mu.$$

a. Prove that ν is a measure.

b. Prove that if g is integrable with respect to ν, then fg is integrable with respect to μ and

$$\int g\,d\nu = \int fg\,d\mu.$$

38. Suppose μ and ν are finite positive measures on the Borel σ-algebra on $[0,1]$ such that $\int f\,d\mu = \int f\,d\nu$ whenever f is real-valued and continuous on $[0,1]$. Prove that $\mu = \nu$.

39. Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$. Let μ_n be a sequence of finite measures on $([0,1], \mathcal{B})$ and let μ be another finite measure on $([0,1], \mathcal{B})$. Suppose $mu_n([0,1]) \to \mu([0,1])$. Prove that the following are equivalent:

a. $\int f\,d\mu_n \to \int f\,d\mu$ whenever f is a continuous real-valued function on $[0,1]$;

b. $\limsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ for all closed $F \subset [0,1]$;

c. $\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ for all open $G \subset [0,1]$;

d. $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for all $A \in \mathcal{B}$ such that $\mu(\partial A) = 0$, where $\partial A \stackrel{\text{def}}{=} \bar{A} - A^o$ is the boundary of A;

e. $\lim_{n\to\infty} \mu_n([0,x]) = \mu([0,x])$ for all $x \in [0,1]$ such that $\mu(\{x\}) = 0$.

40. Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$. Suppose μ_n are finite measures on $([0,1], \mathcal{B})$ such that $\int f\,d\mu_n \to \int_0^1 f(x)\,dx$ whenever f is a real-valued continuous function on $[0,1]$. Suppose that g is a bounded measurable function such that the set of discontinuities of g has measure 0. prove that

$$\int g\,d\mu_n \to \int_0^1 g(x)\,dx.$$

41. Let $\mathcal{B}$ be the Borel σ-algebra on $[0,1]$. Let μ_n be a sequence of finite measures on $([0,1], \mathcal{B})$ such that $\sup_n \mu_n([0,1]) < \infty$. Define $\alpha_n(x) = \mu_n([0,x])$.

a. If $r \in [0,1]$ is rational, prove that there exists a subsequence $\{n_j\}$ such that $\alpha_{n_j}(r)$ converges.

b. Prove that there exists a subsequence $\{n_j\}$ such that $\alpha_{n_j}(r)$ converges for every rational in $[0,1]$.

c. Let $\bar{\alpha}(r) \stackrel{\text{def}}{=} \lim_{n\to\infty} \alpha_n(r)$ for rational $r \in [0,1]$. Prove that $r > s$ with $r, s \in \mathbf{Q} \cap [0,1]$ implies $\bar{\alpha}(r) \leq \bar{\alpha}(s)$.

d. Define

$$\alpha(x) = \lim_{r\to x+,\, r\in\mathbf{Q}} \bar{\alpha}(r).$$

Prove that

$$\alpha(x) = \inf\{\bar{\alpha}(r) : r > x, r \in \mathbf{Q} \cap [0,1]\}.$$

e. Let μ be the Lebesgue-Stieltjes measure associated with α. Prove that

$$\int f\, d\mu_n \to \int f\, d\mu$$

whenever f is a continuous real-valued function on $[0,1]$.