Math 5051
Measure Theory and Functional Analysis I
Homework Assignment 1

Prof. Wickerhauser

Due Friday, September 11th, 2015

Please do Exercises 2, 7, 9, 12, 15, 18*, 20, 21, 22*, 24*, 26, 28, 30, 33*, 35, 36.

Exercises marked with (*) are especially important and you may wish to focus extra attention on those.

You are encouraged to try the other problems in this list as well.

Note: "textbook" refers to "Real Analysis for Graduate Students," version 2.1, by Richard F. Bass. These exercises originate from that source.

1. Find an example of a set X and a monotone class $\mathcal{M}$ consisting of subsets of X such that $\emptyset \in \mathcal{M}$ and $X \in \mathcal{M}$ but $\mathcal{M}$ is not a σ-algebra.

2. Find an example of a set X and two σ-algebras $\mathcal{A}$ and $\mathcal{B}$, each consisting of subsets of X, such that $\mathcal{A} \cup \mathcal{B}$ is not a σ-algebra.

3. Suppose $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots$ are σ-algebras consisting of subsets of a set X. Prove that $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ is necessarily a σ-algebra or else give a counterexample where it is not.

4. Suppose $\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \cdots$ are monotone classes, and put $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i$. Suppose $A_j \uparrow A$ with $A_j \in \mathcal{M}$ for all j. Prove that $A \in \mathcal{M}$ or else give a counterexample where it is not.

5. Let $(Y, \mathcal{A})$ be a measurable space, let X be a set, and suppose that f is a function mapping X into Y, but do not assume that f is injective. Define $\mathcal{B} = \{f^{-1}(A) : A \in \mathcal{A}\}$, a collection of subsets of X. Prove that $\mathcal{B}$ is a σ-algebra.

6. Suppose $\mathcal{A}$ is a σ-algebra with the property that for all nonempty $A \in \mathcal{A}$ there exist nonempty disjoint $B, C \in \mathcal{A}$ with $B \cup C = A$. Deduce that $\mathcal{A}$ must be uncountable.

7. Suppose that $\mathcal{F}$ is a collection of real-valued functions on X such that the constant functions are in $\mathcal{F}$, and that $f+g$, fg, and cf are in $\mathcal{F}$ whenever $f, g \in \mathcal{F}$ and $c \in \mathbf{R}$. Suppose also that $f \in \mathcal{F}$ whenever $f_n \to f$ with each $f_n \in \mathcal{F}$. Finally, define the *characteristic functions*

$$\chi_A(x) \stackrel{\text{def}}{=} \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Prove that $\mathcal{A} \stackrel{\text{def}}{=} \{A \subset X : \chi_A \in \mathcal{F}\}$ is a σ-algebra.

8. Does there exist a σ-algebra that contains countably many elements, but not finitely many elements? (Hint: show that such a σ-algebra must contain infinitely many disjoint elements.)

9. Suppose $(X, \mathcal{A})$ is a measurable space and μ is a nonnegative set function that is finitely additive and such that $\mu(\emptyset) = 0$ and $\mu(B) < \infty$ for some nonempty set $B \in \mathcal{A}$. Suppose further that if $\{A_i : i = 1, 2, \ldots\}$ is any increasing sequence of sets in $\mathcal{A}$, then $\mu(\bigcup_i A_i) = \lim_{i\to\infty} \mu(A_i)$. Show that μ is a measure.

10. Suppose $(X, \mathcal{A})$ is a measurable space and μ is a nonnegative set function that is finitely additive and such that $\mu(\emptyset) = 0$ and $\mu(X) < \infty$. Suppose further that if $\{A_i : i = 1, 2, \ldots\}$ is any sequence of sets in $\mathcal{A}$ that decreases to $\emptyset$, then $\lim_{i\to\infty} \mu(A_i) = 0$. Show that μ is a measure. (Hint: use Exercise 9.)

11. Let X be an uncountable set and let $\mathcal{A}$ be the collection of subsets of X such that either A or A^c is countable. Define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Prove that μ is a measure.

12. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and $A, B \in \mathcal{A}$. Prove that $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$.

13. Prove that if μ_1 and μ_2 are measures on a measurable space, and $\alpha \geq 0$ is a real number, then $\mu \stackrel{\text{def}}{=} \mu_1 + \alpha\mu_2$ is also a measure. (Thus the collection of measures on a measurable space forms a *linear cone*.)

14. Prove that if $\mu_1, \mu_2, \ldots$ are measures on a measurable space, and $\alpha_1, \alpha_2, \ldots \in [0, \infty) \subset \mathbf{R}$, then $\mu \stackrel{\text{def}}{=} \sum_i \alpha_i \mu_i$ is also a measure.

15. Prove that if $(X, \mathcal{A}, \mu)$ is a measure space, $B \in \mathcal{A}$, and we define $\nu(A) = \mu(A \cap B)$ for all $A \in \mathcal{A}$, then ν is a measure.

16. Suppose $\mu_1, \mu_2, \ldots$ are measures on a measurable space $(X, \mathcal{A})$, and $\mu_n(A) \uparrow$ for each $A \in \mathcal{A}$. Define

$$\mu(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_n(A).$$

Is μ a measure? Provide a proof or supply a counterexample.

17. Suppose $\mu_1, \mu_2, \ldots$ are measures on a measurable space $(X, \mathcal{A})$, $\mu_1(X) < \infty$, and $\mu_n(A) \downarrow$ for each $A \in \mathcal{A}$. Define

$$\mu(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_n(A).$$

Is μ a measure? Provide a proof or supply a counterexample. (Hint: compare with Exercise 16.)

18. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $\mathcal{N}$ be the collection of null sets with respect to $\mathcal{A}$ and μ, and let $\mathcal{B} = \sigma(\mathcal{A} \cup \mathcal{N})$.

    a. Show that $B \in \mathcal{B}$ if and only if there exists $A \in \mathcal{A}$ and $N \in \mathcal{N}$ such that $B = A \cup N$.

    b. Define $\bar{\mu}(B) = A$ if $B = A \cup N$ with $A \in \mathcal{A}$ and $N \in \mathcal{N}$. Prove that $\bar{\mu}(B)$ is uniquely defined (also called *well-defined*) on $\mathcal{B}$, that $\bar{\mu}$ is a measure on $\mathcal{B}$, that $(X, \mathcal{B}, \bar{\mu})$ is complete, and that $(X, \mathcal{B}, \bar{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$.

19. Suppose that $X = \mathbf{R}$, $\mathcal{B}$ is the Borel σ-algebra, and μ and ν are two measures on $(X, \mathcal{B})$ such that $\mu((a, b)) = \nu((a, b)) < \infty$ whenever $-\infty < a < b < \infty$. Prove that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$.

20. Suppose that $(X, \mathcal{A})$ is a measurable space and $\mathcal{B}$ is an arbitrary subset of $\mathcal{A}$. Suppose that μ and ν are two σ-finite measures on $(X, \mathcal{A})$ such that $\mu(B) = \nu(B)$ for all $B \in \mathcal{B}$. Is it true that $\mu(A) = \nu(A)$ for all $A \in \sigma(\mathcal{B})$? What if μ and ν are finite measures?

21. Let μ be a measure on the Borel σ-algebra of $\mathbf{R}$ such that $\mu(K) < \infty$ whenever K is compact. Define $\alpha(x) = \mu((0, x])$ if $x \geq 0$ and $\alpha(x) = -\mu((x, 0])$ if $x < 0$. Show that μ is the Lebesgue-Stieltjes measure corresponding to α.

22. Let m be Lebesgue measure and A a Lebesgue measurable subset of $\mathbf{R}$ with $m(A) < \infty$. Let $\epsilon > 0$. Show that there exist a compact set K and an open set G such that $K \subset A \subset G$ and $m(G - K) < \epsilon$.

23. Let X be an uncountable set and define a set function μ on 2^X by

$$\mu(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ 1, & \text{if } A \neq \emptyset \text{ is countable}, \\ 2, & \text{if } A \text{ is uncountable.} \end{cases}$$

Prove that (a) μ is an outer measure, and (b) the only μ-measurable sets are $\emptyset$ and X.

24. Let $(X, \mathcal{A}, \mu)$ be a measure space and define

$$\mu^*(A) = \inf\{\mu(B) : A \subset B, B \in \mathcal{A}\}$$

for all subsets A of X.

a. Show that μ^* is an outer measure.

b. Show that each $A \in \mathcal{A}$ is μ^*-measurable with $\mu^*(A) = \mu(A)$.

25. Let m be Lebesgue-Stieltjes measure corresponding to a right continuous increasing function α. Show that for each x,

$$m(x) = \alpha(x) - \lim_{y \to x-} \alpha(y).$$

26. Suppose m is Lebesgue measure. Define $x + A \stackrel{\text{def}}{=} \{x + y : y \in A\}$ and $cA \stackrel{\text{def}}{=} \{xy : y \in A\}$ for $x \in \mathbf{R}$ and real number c and $A \subset \mathbf{R}$. Show that if A is a Lebesgue measurable set, then $m(x + A) = m(A)$ and $m(cA) = |c|m(A)$.

27. Let m be Lebesgue measure. Suppose that for each n, A_n is a Lebesgue measurable subset of $[0, 1]$. Let B consist of those points x that belong to infinitely many of the A_n.

a. Show that B is Lebesgue measurable.

b. If $m(A_n) > \delta > 0$ for all n, show that $m(B) \geq \delta$.

c. If $\sum_n m(A_n) < \infty$, prove that $m(B) = 0$.

d. Give an example where $\sum_n m(A_n) = \infty$, but $m(B) = 0$.

28. Suppose m is Lebesgue measure. Given $\epsilon \in (0, 1)$, find a Lebesgue measurable set $E \subset [0, 1]$ such that the closure of E is $[0, 1]$ and $m(E) = \epsilon$.

29. If X is a metric space, $\mathcal{B}$ is the Borel σ-algebra, and μ is a measure on $(X, \mathcal{B})$, then the *support* of μ (denoted $\operatorname{supp} \mu$) is the smallest closed set $F \in \mathcal{B}$ such that $\mu(F^c) = 0$. Let $x_n \to x$ be a convergent sequence in $\mathbf{R}$, let $\{\alpha_n\} \subset \mathbf{R}^+$ be summable, and let $\mu \stackrel{\text{def}}{=} \sum_n \alpha_n \delta_{x_n}$. Find the support of μ.

30. Show that if F is a compact subset of $\mathbf{R}$, then then there exists a finite measure on the Borel sets whose support is F. (Hint: see Exercise 29.)

31. Let m be Lebesgue measure. Give an example of a sequence $\{A_i : i = 1, 2, \ldots\}$ of Lebesgue measurable subsets of $[0, 1]$ such that $m(A_i) > 0$ for all i, and $m(A_i \triangle A_j) > 0$ if $i \neq j$, and $m(A_i \cap A_j) = m(A_i)m(A_j)$ for all $i \neq j$. (See p.1 of the textbook for the definition of $\triangle$.)

32. Let $\epsilon \in (0,1)$, let m be Lebesgue measure, and suppose A is a Borel measurable subset of $\mathbf{R}$. Prove that if $m(A \cap I) \leq (1-\epsilon)m(I)$ for every interval I, then $m(A) = 0$. (Hint: see Exercise 22.)

33. Suppose m is Lebesgue measure and A is a Borel measurable subset of $\mathbf{R}$ with $m(A) > 0$. Prove that the *difference set* $B \stackrel{\text{def}}{=} \{x - y : x, y \in A\}$ contains a nonempty open interval centered at the origin. (This is known as the *Steinhaus theorem*.)

34. Let m be Lebesgue measure. Construct a Borel subset A of $\mathbf{R}$ such that $0 < m(A \cap I) < m(I)$ for every open interval I.

35. Let N be the nonmeasurable set defined in Section 4.4 of the text. Prove that if $A \subset N$ and A is Lebesgue measurable, then $m(A) = 0$.

36. Let m be Lebesgue measure. Prove that if A is a Lebesgue measurable subset of $\mathbf{R}$ and $m(A) > 0$, then there is a subset of A that is nonmeasurable.

37. Let X be a set and $\mathcal{A}$ an algebra of subsets of X. Suppose ℓ is a measure on $\mathcal{A}$ such that $\ell(X) < \infty$. Define μ^* using ℓ as in Section 4.1 of the text. Prove that a set A is μ^*-measurable if and only if
$$\mu^*(A) = \ell(X) - \mu^*(A^c).$$

38. Give an example of a set X and a finite outer measure μ^* on subsets $A_n \uparrow A$ of X, and subsets $B_n \downarrow B$ of X, such that $\mu^*(A_n)$ does not converge to $\mu^*(A)$ and $\mu^*(B_n)$ does not converge to $\mu^*(B)$.

39. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and define μ^* as in Exercise 24. Show that if $A_n \uparrow A$ for subsets A_n, A of X, then $\mu^*(A_n) \uparrow \mu^*(A)$. How does this differ from Exercise 38?

40. Suppose A is a Lebesgue measurable subset of $\mathbf{R}$ and
$$B = \bigcup_{x \in A} [x-1, x+1].$$
Prove that B is Lebesgue measurable.