Math 495: Stochastic Processes
Martingale Convergence Example

Prof. Wickerhauser

Tuesday, March 6th, 2018

Abstract. A martingale may be used to model a fair game, but it may also be considered a sequence of successive approximations by simple functions.

1 General Ideas

In the general setting of a probability space Ω with probability measure $\mathbb{P}$ defined on a σ-algebra of subsets $\mathcal{A}$, suppose that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{A}$$

is a filtration of σ-algebras. Then for any $\mathcal{A}$-measurable random variable X, the sequence of conditional expectations

$$X_n \stackrel{\text{def}}{=} E(X|\mathcal{F}_n), \qquad n = 0, 1, 2, \ldots$$

is a martingale consisting of improving approximations to X.

That $\{X_n\}$ is a martingale follows from some very general results:

Lemma 1.1 *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ algebras. Then $\mathcal{A}_1 \cap \mathcal{A}_2$ is a σ-algebra.*

Proof: It is straightforward to check that the intersection contains $\emptyset$, is closed under countable unions and intersections, and is closed under complements. □

Lemma 1.2 *Suppose X is a random variable and $\mathcal{A}_1, \mathcal{A}_2$ are two σ algebras. Then*

$$E(E(X|\mathcal{A}_1)|\mathcal{A}_2) = E(X|\mathcal{A}_1 \cap \mathcal{A}_2).$$

Proof: Suppose that $A \in \mathcal{A}_1 \cap \mathcal{A}_2$. Compute

$$\begin{aligned}
\mathbb{E}(E(E(X|\mathcal{A}_1)|\mathcal{A}_2)\mathbf{1}_A) &= \mathbb{E}(E(X|\mathcal{A}_1)|\mathbf{1}_A) \qquad \text{since } A \in \mathcal{A}_2, \\
&= \mathbb{E}(X\mathbf{1}_A) \qquad \text{since } A \in \mathcal{A}_1, \\
&= \mathbb{E}(E(X|\mathcal{A}_1 \cap \mathcal{A}_2)\mathbf{1}_A) \qquad \text{since } A \in \mathcal{A}_1 \cap \mathcal{A}_2.
\end{aligned}$$

Conclude that the two conditional expectations are equal almost surely. □

Theorem 1.3 *$\{X_n\}$ is a martingale.*

Proof: By the definition of X_n and the previous lemma,

$$E(X_{n+1}|\mathcal{F}_n) = E(E(X|\mathcal{F}_{n+1})|\mathcal{F}_n) = E(X|\mathcal{F}_{n+1} \cap \mathcal{F}_n) = E(X|\mathcal{F}_n) = X_n,$$

since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. □

Now, $\mathbb{E}(|X|) = \mathbb{E}(|X_n|)$ for all $n = 0, 1, 2 \ldots$ since $\Omega \in \mathcal{F}_n$, so if $\mathbb{E}(|X|) < \infty$ then the martingale convergence theorem applies and there is a limit

$$X_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} X_n, \qquad \text{almost surely,}$$

but it may not be that $X_\infty = X$ almost surely as that depends on whether the sequence $\{\mathcal{F}_n\}$ increases to $\mathcal{A}$ in the limit. For example, the nonincreasing constant filtration $\mathcal{F}_n \stackrel{\text{def}}{=} \{\emptyset, \Omega\}$ of minimal σ-algebras may be used to construct a counterexample wherein $X_\infty \neq X$ almost surely.

2 Particular Example

Let Ω be the set $[0, 1)$ considered as a probability space with Lebesgue measure $d\omega$ giving the probability of an event, namely

$$\mathbb{P}(A) = \mathbb{E}(\mathbf{1}_A) = \int_\Omega \mathbf{1}_A(\omega)\, d\omega,$$

where $\mathbf{1}_A$ is the indicator function of the subset $A \subset \Omega$. For example, the open interval $A = (a, b)$ gives

$$\mathbb{P}\left((a, b)\right) = b - a$$

and also

$$\mathbb{P}\left([a, b)\right) = \mathbb{P}\left((a, b]\right) = \mathbb{P}\left([a, b]\right) = b - a.$$

Let $\mathcal{A}$ be the Borel σ-algebra $\mathcal{B}$ of subsets of Ω, namely

$$\mathcal{B} = \sigma\{(a, b) : 0 \leq a < b \leq 1\},$$

where $\sigma\{\mathcal{S}\}$ denotes the smallest σ-algebra that contains the collection of subsets $\mathcal{S}$. Namely, $\mathcal{B}$ contains all countable unions and countable intersections of its elements as well as their complements. Note that $\mathcal{B}$ contains every single-point subset, since $\{\omega\} = \cap_{n=1}^\infty (\omega - \frac{1}{n}, \omega + \frac{1}{n})$ is a countable intersection of open intervals. Thus $[a, b) \in \mathcal{B}$ for every $0 \leq a < b \leq 1$. Likewise, $\mathcal{B}$ contains every countable union of points and intervals.

The probabilities of many events in $\mathcal{B}$ are easy to compute or approximate. In particular, $\mathbb{P}(\{\omega\}) = 0$ for any single point $\omega \in \Omega$, and thus $\mathbb{P}(\cup_{i=1}^\infty \{\omega_i\}) = \sum_{i=1}^\infty \mathbb{P}(\{\omega_i\}) = 0$ for any countable set. Likewise if $B = I_1 \cup I_2 \cup \cdots \subset \Omega$ is a disjoint union of intervals, then $\mathbb{P}(B) = \mathbb{P}(I_1) + \mathbb{P}(I_2) + \cdots \leq 1 = \mathbb{P}(\Omega)$. This summation formula holds even if $I_i \cap I_j$ is a single point for some i, j, so whether the intervals are open, closed, or neither makes no difference.

Now let $\mathcal{F}_0 \stackrel{\text{def}}{=} \sigma(\emptyset) = \{\Omega, \emptyset\}$ be the minimal σ-algebra of subsets of Ω. For $n = 1, 2, \ldots,$ put

$$\mathcal{F}_n \stackrel{\text{def}}{=} \sigma\left\{ I_{nk} \stackrel{\text{def}}{=} \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) : 0 \leq k < 2^n \right\}.$$

This is a finite[1] σ-algebra (for fixed n) consisting of disjoint unions of the *minimal intervals* I_{nk} which have the property that if $S \in \mathcal{F}_n$ is nonempty and $S \subset I_{nk}$, then $S = I_{nk}$. It is obvious that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{B}$, so the collection $\{\mathcal{F}_n\}$ is a filtration of the probability space $(\Omega, \mathcal{B}, \mathbb{P})$.

[1] As an exercise, show that $\mathcal{F}_n$ contains 2^{2^n} elements.

To define a martingale of successive approximations, start with a given continuous function $f : \Omega \to \mathbf{R}$ for the random variable. A sequence of successive approximations to f may be constructed using conditional expectation:

$$f_n \stackrel{\text{def}}{=} E(f|\mathcal{F}_n), \qquad n = 0, 1, 2, \ldots$$

As in the first section above, $\{f_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$. If $\mathbb{E}(|f|) < \infty$, then the martingale convergence theorem applies and one may ask whether $\lim_{n\to\infty} f_n = f$. In this case the limit may be computed explicitly.

The conditional expectation f_n is a random variable that is measurable with respect to $\mathcal{F}_n$, which means that for every $x \in \mathbf{R}$, the inverse image $f_n^{-1}(x) \in \mathcal{F}_n$ is a finite union of minimal intervals $I_{nk} \stackrel{\text{def}}{=} [\frac{k}{2^n}, \frac{k+1}{2^n})$ of $\mathcal{F}_n$ on which f_n takes the constant value x. But for any such minimal interval I_{nk} where f_n takes that constant value, the conditional expectation property allows it to be evaluated:

$$2^{-n}x = \mathbb{E}(x\mathbf{1}_{I_{nk}}) = \mathbb{E}(f_n\mathbf{1}_{I_{nk}}) = \mathbb{E}(f\mathbf{1}_{I_{nk}}) = \int_{I_{nk}} f(\omega)\,d\omega = \int_{k/2^n}^{(k+1)/2^n} f(\omega)\,d\omega.$$

Write f_{nk} for the value that f_n takes on minimal interval I_{nk} to get a finite expansion for f_n:

$$f_n(\omega) = \sum_{k=0}^{2^n-1} f_{nk}\mathbf{1}_{I_{nk}}(\omega); \qquad f_{nk} \stackrel{\text{def}}{=} 2^n \int_{k/2^n}^{(k+1)/2^n} f(\omega)\,d\omega,$$

so f_n is a piecewise constant function with jump discontinuities at rational points with denominator 2^n. (Such points are called *dyadic* rationals.)

To show convergence of this martingale to the original f, fix $z \in [0, 1)$ and let $\epsilon > 0$ be given. Since f is continuous, there is some $\delta > 0$ such that $|\omega - z| < \delta$ implies $|f(\omega) - f(z)| < \epsilon$. Choose a positive integer N such that $2^{-N} < \delta$ and suppose that $n \geq N$. There is a unique interval I_{nk} containing z, and it satisfies

$$\omega \in I_{nk} \implies |z - \omega| < \delta \implies |f(z) - f(\omega)| < \epsilon,$$

from which it follows that

$$|f(z) - f_n(z)| \leq 2^n \int_{I_{nk}} |f(z) - f(\omega)|\,d\omega < \epsilon.$$

Conclude that $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in \Omega$.