Math 495: Stochastic Processes
Final Examination

Name:_____

Due Friday, 4 May 2018, by 8:00pm

6 Problems on 1+6 Pages

You may use the textbook, your notes, the model homework solutions, and any other existing written reference materials and aids to computation, but you must find solutions by yourself, without collaboration.

1. Let X_n be the number of individuals in the nth generation of a branching process in which each individual produces offspring from a distribution with finite mean μ and finite variance. Prove that $M_n \stackrel{\text{def}}{=} \mu^{-n} X_n$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}$.

2. Suppose that Y is a nonnegative random variable with $\mathbb{E}(Y) < \infty$. Let $\{X_n\}$ be any sequence of random variables and assume that $|X_n| \leq Y$ for all n. Prove that $\{X_n\}$ must be uniformly integrable.

3. Give an example of a renewal process with $\mathbb{E}(T_i) = \mu < \infty$ such that large-time residual life has infinite mean.

4. Let $\{W_t\}$ be standard Brownian motion starting from $W_0 = 0$. Fix $t > 0$. Prove that for any $a > 0$,

$$\mathbb{P}(W_t \geq a) \leq \sqrt{\frac{2t}{\pi a^2}}\, e^{-a^2/2t}.$$

5. Let X_t be standard Brownian motion and put

$$M = M(T) = \max\{X_t : 0 \leq t \leq T\}$$

Find the expectation and variance of M as a function of T.

6. Suppose that $d > 1$ and W_t is standard Brownian motion in $\mathbf{R}^d$ starting from $W_0 = (0, \ldots, 0)$. Let $f : \mathbf{R}^d \to \mathbf{R}$ be the function

$$f(x_1, \ldots, x_d) = x_1 \cdots x_d = \prod_{i=1}^{d} x_i.$$

(a) Use Itô's formula to prove that $M_t \stackrel{\text{def}}{=} f(W_t)$ is a martingale.

(b) Without using the optional sampling theorem, explain why $\mathbb{E}(M_T) = M_0$ for any stopping time T.