# Math 456: Introduction to Financial Mathematics

## Homework Set 6

1. Suppose that $C$ and $P$ are European-style Call and Put options, respectively, at strike price $K$ and expiry $T$, for a risky underlying asset $S$ with spot price $S_0$. Show that $0 < C(0) \leq S_0$ and $0 < P(0) \leq K$. (Hint: construct an arbitrage otherwise.)

   **Solution:** First note that $C(0) > 0$ and $P(0) > 0$ are consequences of Theorem 1.1.

   Next, suppose that $C(0) > S_0$. Sell one $C$ and buy one $S$ at time $t = 0$, keeping the surplus $C(0) - S_0 > 0$. At expiry $T$, if the buyer exercises $C$, collect $K > 0$ in exchange for $S$. Otherwise, sell $S$ for $S(T) > 0$.

   Finally, suppose that $P(0) > K$. Sell one $P$ for $P(0)$ at $t = 0$ and keep the proceeds. At expiry $t = T$, if the buyer exercises $P$, receive that $S$ at strike price $K$, leaving a surplus $P(0) - K > 0$ and sell it for $S(T) > 0$, netting additional profit. Otherwise, keep the original $P(0)$ with no further obligations.

   In all cases there is a positive payoff with no initial investment. Conclude by the no-arbitrage axiom that $0 < C(0) \leq S_0$ and $0 < P(0) \leq K$. □

2. Show that the Eq.7.11 and Eq.7.12 probabilities produce the Arrow-Debreu spot prices $\lambda(n, j)$ in Eq.7.10 using Jamshidian's forward induction, Eq.3.21 on p.69.

   **Solution:** This may be proved by induction on $n$. For $n = 0$, the only Arrow-Debreu spot price is $\lambda(0, 0) = 1$, so it agrees with the value $Q(0, 0)/R^0 = 1$ from Jackwerth's construction.

   Now suppose that the values $\{\lambda(n-1, j) : j = 0, 1, \ldots, n-1\}$ produced by Jamshidian's induction agree with Jackwerth's values and that $p$ and $1 - p$ are given by Eq.7.11 and Eq.7.12, respectively. (As usual, take $\lambda(n, j) = 0$ if $j < 0$ or $j > n$.) Compute $\lambda(n, j)$ for $j = 0, 1, \ldots, n$ by substituting the expressions from Jackwerth's construction into Eq.3.21:

$$\begin{aligned}
\lambda(n,j) &= \frac{1-p(n-1,j)}{R}\lambda(n-1,j) \;+\; \frac{p(n-1,j-1)}{R}\lambda(n-1,j-1) \\
&= \frac{1}{R}\left(1 - w\left(\frac{j}{n}\right)\right)\left[\frac{Q(n,j)}{Q(n-1,j)}\right]\frac{Q(n-1,j)}{R^{n-1}} \\
&\qquad + \frac{1}{R}w\left(\frac{j}{n}\right)\left[\frac{Q(n,j)}{Q(n-1,j-1)}\right]\frac{Q(n-1,j-1)}{R^{n-1}} \\
&= \left[\left(1 - w\left(\frac{j}{n}\right)\right) + w\left(\frac{j}{n}\right)\right]\frac{Q(n,j)}{R^n} \;=\; \frac{Q(n,j)}{R^n}.
\end{aligned}$$

   Thus Jamshidian's forward induction with Jackwerth's probabilities produces Jackwerth's Arrow-Debreu spot prices. □

3. Compute implied volatility for the data in Table 7.1 using both Black-Scholes and CRR with $N = 20$. Tabulate and compare the results.

   **Solution:** Use the Octave code that produced Figure 7.1, with the following modifications:

```
Ks=40:46; Ts=[3,9,21,35,63]; % strikes and days to expiry
C=[4.14 4.35 4.85 4.60 5.15; ...
   3.20 3.42 3.50 4.00 4.35; ...
   2.24 2.72 2.89 3.15 3.65; ...
   1.30 1.51 1.96 2.44 2.87; ...
   0.63 0.86 1.34 1.71 2.39; ...
   0.23 0.45 0.84 1.24 1.76; ...
   0.07 0.22 0.52 0.88 1.39]; % Call premiums on 2021-12-14
S0=44.13; r=0.05; % spot price and riskless APR on 2021-12-14
VimpBS=zeros(length(Ks),length(Ts)); % BS implied volatilities
VimpCRR=zeros(length(Ks),length(Ts)); % CRR implied volatilities
minv=0.01; maxv=0.99; tol=0.00001; % bisection parameters
for col=1:length(Ts)
  T=Ts(col)/365; % time to expiry in years
  for row=1:length(Ks)
    f=@(v) BS(T,S0,Ks(row),r/100,v); % Black-Scholes Call
    VimpBS(row,col)=bisection(f,C(row,col),minv,maxv,tol);
    g=@(v) CRReur(T,S0,Ks(row),r/100,v,20)(1,1); % CRR Call
    VimpCRR(row,col)=bisection(g,C(row,col),minv,maxv,tol);
  end
end
```

The output is in Table 1:

| | Dec17 | | Dec23 | | Jan7 | | Jan21 | | Feb18 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $K$ | $CRR$ | $BS$ | $CRR$ | $BS$ | $CRR$ | $BS$ | $CRR$ | $BS$ | $CRR$ | $BS$ |
| 40 | .50 | .50 | .54 | .54 | .54 | .55 | .36 | .35 | .37 | .37 |
| 41 | .56 | .56 | .49 | .49 | .35 | .35 | .39 | .39 | .35 | .35 |
| 42 | .47 | .47 | .52 | .52 | .39 | .39 | .36 | .36 | .34 | .34 |
| 43 | .36 | .36 | .30 | .30 | .31 | .32 | .33 | .34 | .31 | .31 |
| 44 | .35 | .35 | .29 | .29 | .30 | .30 | .30 | .30 | .32 | .32 |
| 45 | .34 | .35 | .29 | .29 | .28 | .29 | .29 | .30 | .29 | .29 |
| 46 | .36 | .36 | .30 | .30 | .28 | .29 | .30 | .30 | .29 | .29 |

Table 1: Implied volatilities by CRR ($N = 20$) and Black-Scholes methods, from December 14, 2021 closing prices for BAC American-style Call options at $r = 0.05\%$ and $S_0 = \$44.13$.

**Remark.** As expected, the two methods produce nearly identical values. To further check the results and the code, compare the Black-Scholes ($BS$) values with the $\sigma$ columns of Table 7.1. □

4. The table below gives part of the options chain for American-style Calls on Bank of America common stock (BAC) as of closing on March 17, 2022, when the spot price was \$43.03:

| Strike price $K$ | T= 1 d (3/18) | 8 d (3/25) | 15 d (4/01) | 21 d (4/08) | 27 d (4/14) |
|---|---|---|---|---|---|
| 42.00 | 1.10 | 1.44 | 1.76 | 1.96 | 2.18 |
| 43.00 | 0.34 | 0.86 | 1.10 | 1.33 | 1.58 |
| 44.00 | 0.06 | 0.44 | 0.69 | 0.88 | 1.06 |
| 45.00 | 0.02 | 0.18 | 0.35 | 0.53 | 0.71 |
| 46.00 | 0.01 | 0.07 | 0.16 | 0.31 | 0.44 |
| 47.00 | 0.01 | 0.02 | 0.09 | 0.17 | 0.27 |

Also, the US T-bill rates for various maturities were

```
Date        4 wk    8 wk   13 wk   26 wk   52 wk
03/14/2022  0.22    0.30    0.45    0.84    1.20
03/15/2022  0.21    0.29    0.46    0.84    1.19
03/16/2022  0.23    0.28    0.43    0.84    1.26
03/17/2022  0.20    0.30    0.40    0.79    1.20
```

Use this data to compute and plot the volatility surface for BAC.

**Solution:** Use the commands that produced Figure 7.1, but with the data for this problem. Set the riskless rate for all calculations to be the 4-week rate averaged over the 4 days sampled, which is 0.215% APR.

```
Ks=42:47; Ts=[1,8,15,21,27]; % strikes and days to expiry
C=[1.10 1.44 1.76 1.96 2.18 ;...
   0.34 0.86 1.10 1.33 1.58 ;...
   0.06 0.44 0.69 0.88 1.06 ;...
   0.02 0.18 0.35 0.53 0.71 ;...
   0.01 0.07 0.16 0.31 0.44 ;...
   0.01 0.02 0.09 0.17 0.27 ]; % Call premiums on 2022-03-17
S0=43.03; r=0.215; % spot price and riskless APR
Vimp=zeros(length(Ks),length(Ts)); % implied volatilities
minv=0.01; maxv=0.99; tol=0.00001; % bisection parameters
for col=1:length(Ts)
 T=Ts(col)/365; % time to expiry in years
 for row=1:length(Ks)
   f=@(v) BS(T,S0,Ks(row),r/100,v); % Black-Scholes Call
   Vimp(row,col)=bisection(f,C(row,col),minv,maxv,tol);
 end
end
mesh(Ts,Ks,Vimp); % note the transposed order (T,K,V(K,T))
title("Implied Volatility Surface");
xlabel("T (days)"); ylabel("K");
```

The results may be seen in Figure 1. □

5. Suppose that a share of XYZ has a spot price of \$47.12, that riskless interest rates for the next month are expected to be a constant 0.66% APR, and that the premiums for European-style Call options expiring in 4 weeks ($T = 4/52$) are as follows:

```
Strike price:  45.00  46.00  47.00  48.00  49.00
Call premium:   3.52   2.78   2.10   1.44   1.37
```

(a) Construct an implied binomial tree for these inputs using Rubinstein's 1-2-3 algorithm. Display it along with the implied risk neutral up probabilities.

(b) Plot the three weight functions $w_1(x) = \sqrt{x}$, $w_2(x) = x^2$, and $w_3(x) = (1 - \cos(\pi x))/2$, for $0 \le x \le 1$, on the same graph.

(c) Apply Rubinstein's 1-2-3 algorithm with Jackwerth's generalization to the data, using weights $w_1, w_2, w_3$ from part (b). Compare $S$, $p$, and $Q$ for the three weights.

**Solution:**

(a) Apply Rubinstein's original 1-2-3 algorithm using `IBT123J.m` with the following Octave commands:

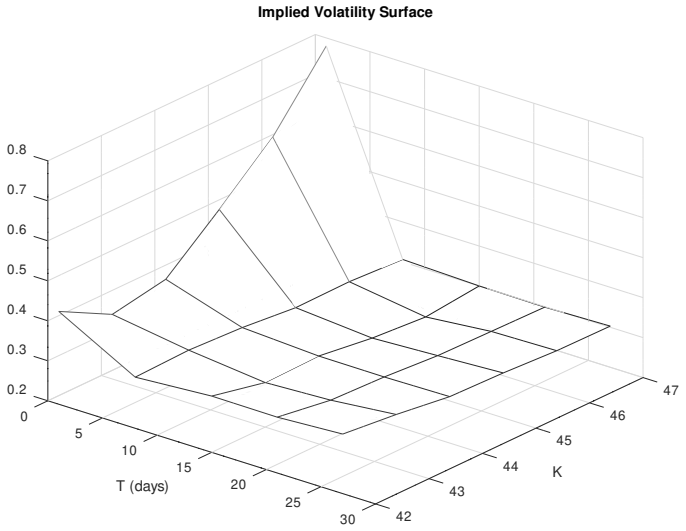



Figure 1: Graph from Exercise 4.

```
Ks=[45,46,47,48,49]; Cs=[3.52 2.78 2.10 1.44 1.37];
S0=47.12; r=0.66/100; rho=exp(r); w=@(x)x;
[S,Q,up,down,pu,N1]=IBT123J(S0,Ks,Cs,rho,w); S,pu
```

That produces the following output, edited for space savings:

```
S =  % implied binomial tree   pu = % risk neutral up probs.
47.12                          0.5411
43.48 50.36                    0.3710 0.6853
41.14 47.63 51.73              0.0639 0.8916 0.5905
40.88 46.11 47.90 54.53        0.0559 0.1818 0.9779 0.3218
40.64 46.00 47.00 48.00 68.57
```

(b) The three weight functions all differ from $w(x) = x$: $w_1(x) = \sqrt(x)$ is concave, $w_2(x) = x^2$ is convex, and $w_3(x) = (1 - \cos(\pi x))/2$ has a unique inflection point. They may be plotted as in Figure 2 using the following Octave/MATLAB commands:

```
w1=@(x)sqrt(x); w2=@(x)x.^2; w3=@(x)(1-cos(pi*x))/2;
t=0:0.01:1; plot(t,w1(t),"r--",t,w2(t),"b..",t,w3(t),"k-");
legend("w1","w2","w3","location","southeast");
title("Weight Functions for Jackwerth's Generalization");
```

(c) Run three experiments as follows, reusing previously assigned variables from parts (a) and (b):

```
[S1,Q1,up,down,pu1,N1]=IBT123J(S0,Ks,Cs,rho,w1); S1,Q1,pu1
[S2,Q2,up,down,pu2,N1]=IBT123J(S0,Ks,Cs,rho,w2); S2,Q2,pu2
[S3,Q3,up,down,pu3,N1]=IBT123J(S0,Ks,Cs,rho,w3); S3,Q3,pu3
```

The results, side-by-side, are:

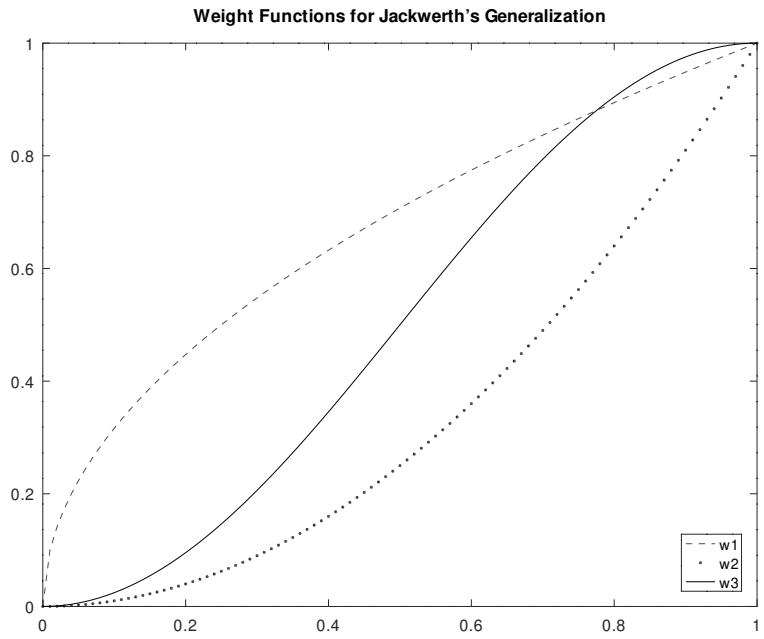



Figure 2: Weight functions for Jackwerth's generalization in Rubinstein's 1-2-3 algorithm: $w_1(x) = \sqrt(x)$, $w_2(x) = x^2$, and $w_3(x) = (1 - \cos(\pi x))/2$.

```
S1 =                     S2 =                     S3 =
47.12                    47.12                    47.12
44.57 51.57              41.74 49.74              43.82 50.53
41.50 47.76 53.72        40.73 47.31 50.58        40.98 47.63 52.87
41.14 46.24 47.91 57.57  40.65 46.01 47.88 52.30  40.76 46.09 47.90 57.11
% ...each last S row is 40.64,  46.00,  47.00,  48.00,  68.57
Q1 =                     Q2 =                     Q3 =
1                        1                        1
.6246 .3754              .3182 .6818              .4963 .5037
.3110 .4436 .2455        .2657 .2100 .5243        .2793 .4340 .2867
.2853 .0444 .5202 .1500  .2589 .0617 .3492 .3303  .2639 .0616 .5170 .1574
% ...each last Q row is  0.2551,  0.0604,  0.0201,  0.5939,  0.0705
pu1 =                    pu2 =                    pu3 =
.3754                    .6818                    .5037
.5021 .6539              .1650 .7690              .4372 .5692
.0825 .9577 .6111        .0258 .7390 .6300        .0551 .8935 .5492
.1058 .3204 .9887 .4697  .0146 .0816 .9568 .2133  .0335 .1634 .9805 .4476
```

Note that there is one less row for the up probabilities. □

6. Prove that the intersection of any collection of convex sets is convex.

**Solution:** Suppose that $\{C_\alpha : \alpha \in I\}$ is an arbitrary collection of convex sets. Let $S = \cap_{\alpha \in I}$ be the intersection of all of them.

If $S = \emptyset$, then $S$ is convex as there is nothing to check.

Otherwise, suppose $x, y \in S$, fix $t \in [0,1]$, and let $z = tx + (1-t)x$. For every $\alpha \in I$, $x, y \in C_\alpha$ implies $z \in C_\alpha$, since $C_\alpha$ is convex. But then $z \in S = \cap_\alpha C_\alpha$. Conclude that $S$ is convex. □

7. Suppose that a market model has five states, a riskless asset returning $R = 1.02$, and two risky assets $a, b$ with spot prices $a_0 = 20$ and $b_0 = 12$ and payoffs $\mathbf{a} = (10, 15, 20, 25, 30)$ and $\mathbf{b} = (17, 15, 12, 10, 7)$, respectively.

   (a) Prove that the model is arbitrage-free.

   (b) Find the no-arbitrage bid-ask interval for a European-style Call option on $a$ with strike price 20.

   (c) Find the no-arbitrage bid-ask interval for a European-style Put option on $b$ with strike price 13.

   **Solution:** The computations may be done in Octave with the `glpk` package. Begin by putting the data into a finite market model $A\mathbf{q}$:

```
a=[10,15,20,25,30]; b=[17,15,12,10,7]; a0=20; b0=12;
R=1.02; A=[R R R R R; a; b]; q=[1; a0; b0]; A,q
```

   (This is in the notation of Definition 4.)

   (a) To show that $A, \mathbf{q}$ is arbitrage-free, by Fundamental Theorem 8.4 it suffices to find $\mathbf{k} > \mathbf{0}$ such that $\mathbf{q} = A\mathbf{k}$. This $3 \times 5$ linear system may be placed into row echelon form, yielding

```
  rref([A R*q])
%  1.00000   0.00000   0.00000  -1.00000  -1.00000  -0.48000
%  0.00000   1.00000   0.00000   1.00000  -0.00000   0.88000
%  0.00000   0.00000   1.00000   1.00000   2.00000   0.60000
```

   (Use `R*q` in the augmented matrix to get the risk neutral probabilities $\mathbf{p} = R\mathbf{k}$, which sum to 1, instead of the discounted vector $\mathbf{k}$ which will sum to $\frac{1}{R}$.) The complete set of three pivot rows shows that the system is consistent but underdetermined and thus has a two-parameter family of solutions. By Corollary 8.8, it suffices to check that the unique minimal norm solution is positive. Octave computes it with

```
A\q*R % p =  0.024  0.432  0.040  0.448  0.056
A\q   % k = 0.023529  0.423529  0.039216  0.439216  0.054902
```

   Conclude that $A, \mathbf{q}$ is arbitrage-free.

   **Remark.** The row echelon form of `[A R*q]` shows how to find the complete solution set of positive vectors $R\mathbf{k} \stackrel{\text{def}}{=} \mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$. Identify the pivot variables $p_1, p_2, p_3$, so $p_4$ and $p_5$ are free variables. The general solution may be expressed as

$$\begin{aligned} p_1 &= p_4 + p_5 - 0.48 \\ p_2 &= -p_4 + 0.88 \\ p_3 &= -p_4 - 2p_5 + 0.60 \end{aligned}$$

   (Notice that $\sum_i p_i = 1$.) To find a positive solution $\mathbf{p} > \mathbf{0}$ requires solving the simultaneous inequalities

$$p_4 > 0,\ p_5 > 0,\ p_4 + p_5 > 0.48,\ p_4 < 0.88,\ p_4 + 2p_5 < 0.60,$$

   which reduce to the intervals $0 < p_5 < 0.12$ and $0.48 - p_5 < p_4 < 0.60 - 2p_5$. One of the infinitely many solutions is thus parametrized by the midpoints:

```
p5=0.12/2; p4=((0.48-p5)+(0.60-2*p5))/2;
p1=p4+p5-0.48; p2= -p4+0.88; p3= -p4-2*p5+0.60;
p=[p1 p2 p3 p4 p5] % p = 0.03  0.43  0.03  0.45  0.06
k=p/R % k = 0.029412  0.421569  0.029412  0.441176  0.058824
```

This gives another explicit positive solution $\mathbf{k} > \mathbf{0}$, proving that $A, \mathbf{q}$ is arbitrage-free without using Corollary 8.8.

For parts (b) and (c), reuse the Octave code from Section 8.2.3, first putting the parameters, market matrix and spot prices into GLPK format:

```
sellctype="LLLLL"; % Lower constraint type A'*x(j)>=bb(j), j=1:5
sellsense=1; % Optimization direction for q'*x: "1" ==> "min"
buyctype="UUUUU"; % Upper constraint type A'*x(j)=<bb(j), j=1:5
buysense=-1; % Optimization direction for q'*x: "-1" ==> "max"
vartype="CCC";   % ==> x(j) is Continuous, j=1:3
param.msglev=1;  % ==> use a low verbosity level
huge=1000; infty=huge*(abs(a0)+abs(b0)); % huge and huger
param.itlim=huge; % ==> huge maximum number of iterations
lb=[-infty; -huge; -huge]; % huge Lower bounds on x
ub=[ infty;  huge;  huge]; % huge Upper bounds on x
```

(b) Compute the Call payoff on asset $a$, then find a superreplication and a subreplication, taking the derivative seller's and buyer's perspectives, respectively:

```
Ka=20; ba=max(a-Ka,0); % C(T): payoff for "a" Call
[hs,ask]=glpk(q,A',ba,lb,ub,sellctype,vartype,sellsense,param)
% hs = [102.9412; -2.00; -5.00]; Cost-minimizing hedge portfolio
% ask = 2.9412;  minimum cost to superreplicate the Call
[hb,bid]=glpk(q,A',ba,lb,ub,buyctype,vartype,buysense,param)
% hb = [ -117.6471; 3.00; 5.00]; Cost-maximizing hedge portfolio
% bid = 2.3529;  maximum cost to subreplicate the Call
```

Since bid is strictly less than ask, by Corollary 8.9 there is no exact hedge for this derivative in this market.

(c) Compute the Put payoff on asset $b$, then find a superreplication and a subreplication, taking the derivative seller's and buyer's perspectives, respectively:

```
Kb=13; bb=max(Kb-b,0); % P(T): payoff for "b" Put
[hs,ask]=glpk(q,A',bb,lb,ub,sellctype,vartype,sellsense,param)
% hs = [ 61.7647; -1.20; -3.00]; Cost-minimizing hedge portfolio
% ask = 1.7647;  minimum cost to superreplicate the Call
[hb,bid]=glpk(q,A',bb,lb,ub,buyctype,vartype,buysense,param)
% hb = [ -26.47059; 0.80; 1.00]; Cost-maximizing hedge portfolio
% bid = 1.5294;  maximum cost to subreplicate the Call
```

Since bid is strictly less than ask, by Corollary 8.9 there is no exact hedge for this derivative in this market. □

8. Suppose that $n > 2$ and market model $A, \mathbf{q}$ has

$$A = \begin{pmatrix} R & \cdots & R \\ a_1 & \cdots & a_n \end{pmatrix},$$

where $R > 1$ is the riskless return and $\mathbf{a} = (a_1, \ldots, a_n)$ is a nonconstant payoff vector for the sole risky asset.

(a) Find necessary and sufficient conditions on $\mathbf{q}$ such that $A, \mathbf{q}$ is arbitrage-free. (Hint: use the Fundamental Theorem.)

(b) Exhibit a derivative payoff $\mathbf{d}$ for which no exact hedge exists. (This shows that $A$ is not a complete market.)

(c) Exhibit a derivative $\mathbf{d}$ for which an exact hedge does exist.

**Solution:** (a) Following the hint, observe that $A, \mathbf{q}$ is arbitrage-free if and only if there exists a positive vector $\mathbf{k} = [k_1, \ldots, k_n]^T$ such that $\mathbf{q} = A\mathbf{k}$. But $\mathbf{q} = [1, S_0]^T$, so $R\mathbf{k}$ is a p.d.f., so if $\mathbf{q} = A\mathbf{k}$, then

$$S_0 = \mathbf{q}(2) = A\mathbf{k}(2) = \sum_{i=1}^{n} a_i k_i = \frac{1}{R} \sum_{i=1}^{n} a_i (R k_i),$$

which is possible if and only if $\min\{a_i\} < RS_0 < \max\{a_i\}$.

**Remark.** Thus $RS_0$ must be inside the range of the payoffs $\{a_i\}$.

(b) It may be assumed that $a_1 < a_2 \leq a_3$. (Otherwise, simply renumber the states.) Submatrix

$$A_2 = \begin{pmatrix} R & R \\ a_1 & a_2 \end{pmatrix}$$

is invertible, so the numbers $h_0, h_1$ are uniquely determined by

$$\begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = A_2^{-1} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where $\mathbf{d} = [d_1, d_2, d_3, \ldots]^T$ is the payoff vector for the derivative to be hedged. Now choose $d_1 = d_2 = 1$, determine $h_0, h_1$, and choose $d_3$ different from $h_0 R + h_1 a_3$. The derivative with this payoff cannot be exactly hedged in this market.

(c) As in part (b), choose $d_1 = d_2 = 1$, determine $h_0, h_1$, but choose

$$d_i = h_0 R + h_1 a_i, \qquad i = 3, \ldots, n.$$

The row vector $\mathbf{d} = [1, 1, d_3, \ldots, d_n]$ lies in the row space of $A$ and thus is exactly hedged by $\mathbf{h} = [h_0, h_1]$. □