Math 456: Introduction to Financial Mathematics

Homework Set 2

1. Let
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$
be the cumulative distribution function of the standard normal random variable. Prove that
$$\Phi(x) + \Phi(-x) = 1$$
for every x.

Solution: This result holds for every cumulative distribution function with a density function ϕ that is symmetric about $t = 0$, namely for which $\phi(-t) = \phi(t)$ for all t. In particular, that is true for $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, since $(-t)^2 = t^2$:
$$\begin{aligned}
\Phi(x) + \Phi(-x) &= \int_{-\infty}^{x} \phi(t)\, dt + \int_{-\infty}^{-x} \phi(t)\, dt \\
&= \int_{-\infty}^{x} \phi(t)\, dt + \int_{x}^{\infty} \phi(-t)\, dt \\
&= \int_{-\infty}^{x} \phi(t)\, dt + \int_{x}^{\infty} \phi(t)\, dt \\
&= \int_{\infty}^{\infty} \phi(t)\, dt \;=\; 1,
\end{aligned}$$
where the second step employs the change of variable $t \leftarrow -t$, the third step follows from symmetry $\phi(-t) = \phi(t)$, and the last equality holds since ϕ is a p.d.f. □

2. (a) There is an error in the textbook at Eq.2.7 on p.26. Find the correct mean and variance of the endpoint $X(n, \cdot)$ using Eq.2.6 on p.24 and the known mean and variance
$$\mathrm{E}(B) = np, \qquad \mathrm{Var}(B) = np(1-p),$$
where B is the binomial random variable taking values $k \in \{0, 1, \ldots, n\}$, with k being the number of successes in n independent trials with success probability p.

(b) Renormalize $X(t,\omega)$ to get the continuous piecewise linear stochastic process $\hat{X}(t,\omega)$ on $0 \le t \le 1$ defined by:

$$\hat{X}(t,\omega) \stackrel{\text{def}}{=} \frac{1}{\sqrt{n}} X(nt,\omega).$$

Compute the mean and variance of $\hat{X}(1,\omega)$.

Solution: (a) Since B gives the number k of successes, and $X = -n+2k$, we may write $X = -n + 2B$. Since the success, or up, probability for our random walk is equal to the failure, or down, probability, we have $p = 1/2$. Applying Eq.2.6 gives

$$\mathrm{E}(X) = -n + 2\mathrm{E}(B) = -n + 2(np) = 0,$$

and

$$\mathrm{Var}(X) = 2^2 \mathrm{Var}(B) = 2^2 np(1-p) = n,$$

using the given formulas for the standard binomial random variable.

(b) In particular,

$$\hat{X}(1,\omega) = \frac{1}{\sqrt{n}} X(n,\omega)$$

By Eq.2.6 and the corrected Eq.2.7 from part a,

$$\mathrm{E}(\hat{X}(1,\cdot)) = \mathrm{E}(X(n,\cdot)) = 0,$$

and

$$\mathrm{Var}(\hat{X}(1,\cdot)) = \left(\frac{1}{\sqrt{n}}\right)^2 \mathrm{Var}(X(n,\cdot)) = 1.$$

□

3. Use the Call-Put Parity formula to derive the Black-Scholes formula for European-style Put options, Eq.2.26:

$$P(0) = e^{-rT} K \Phi(-d_2) - S_0 \Phi(-d_1).$$

Solution: Under the assumptions of no expected arbitrage and constant riskless rate r, the Call-Put Parity Formula is

$$C(0) - P(0) = S_0 - K e^{-rT},$$

so, using the Black-Scholes formula for call premiums,

$$\begin{aligned} P(0) &= C(0) - S_0 + K e^{-rT} \\ &= K e^{-rT}[1 - \Phi(d_2)] - S_0[1 - \Phi(d_1)] \\ &= K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1), \end{aligned}$$

using the identity $\Phi(-x) = 1 - \Phi(x)$ from problem 1 above. □

4. Let d_1, d_2 be defined as in Eq.2.24, and let ϕ be the standard normal p.d.f. defined in Eq.2.18. Show that

$$S_0\phi(d_1) - Ke^{-rT}\phi(d_2) = 0.$$

(Hint: Notice that $d_1 - d_2 = v\sqrt{T}$ and

$$d_1 + d_2 = 2(\log\frac{S_0}{K} + rT)/(v\sqrt{T}),$$

and thus $(d_1^2 - d_2^2)/2 = (d_1 - d_2)(d_1 + d_2)/2 = \log(S_0/K) + rT$.)

Solution: Expand $\sqrt{2\pi}\left[S_0\phi(d_1) - Ke^{-rT}\phi(d_2)\right]$ to get

$$S_0e^{-d_1^2/2} - Ke^{-rT}e^{-d_2^2/2} = e^{-d_2^2/2}\left(S_0e^{(d_2^2-d_1^2)/2} - Ke^{-rT}\right).$$

Now $(d_2^2 - d_1^2)/2 = -\log(S_0/K) - rT$ from the hint, so

$$S_0\exp(-\log\frac{S_0}{K} - rT) - Ke^{-rT} = S_0\frac{K}{S_0}e^{-rT} - Ke^{-rT} = 0,$$

proving the result. □

5. Derive Δ_C and Δ_P in Section 2.3.3 by differentiating the Black-Scholes formulas.

Solution: First, differentiate the Black-Scholes $C(0)$ formula, Eq.2.25, with respect to S_0:

$$\Delta_C = \frac{\partial C(0)}{\partial S_0} = \Phi(d_1) + S\Phi'(d_1)\frac{\partial d_1}{\partial S_0} - Ke^{-rt}\Phi'(d_2)\frac{\partial d_2}{\partial S_0},$$

by the product rule and the chain rule. Also, from Eq.2.24,

$$\frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0} = \frac{1}{S_0v\sqrt{T}}.$$

Next, by the Fundamental Theorem of Calculus,

$$\Phi'(d_1) = \phi(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}, \quad \Phi'(d_2) = \phi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-d_2^2/2},$$

where $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the standard normal density. These may be combined to give

$$\Delta_C = \Phi(d_1) + \frac{S_0\phi(d_1) - Ke^{-rT}\phi(d_2)}{S_0v\sqrt{T}}.$$

But $S_0\phi(d_1) - Ke^{-rT}\phi(d_2) = 0$ by Exercise 4, so the second term vanishes. Conclude that $\Delta_C = \Phi(d_1)$, as claimed.

Finally, use the Call-Put Parity formula to find Δ_P:

$$C(0) - P(0) = S_0 - Ke^{-rT} \quad \Longrightarrow \; P(0) = C(0) - S_0 + Ke^{-rT}.$$

The partial derivative with respect to S_0 yields $\Delta_P = \Delta_C - 1 = \Phi(d_1) - 1$, also as claimed. □

6. Derive Γ_C and Γ_P in Section 2.3.3 from Black-Scholes.

   Solution: Compute Gammas by differentiating the Deltas with respect to S_0. Using Eq.2.28 for Δ_C,

$$\Gamma_C = \frac{\partial \Delta_C}{\partial S_0} = \frac{\partial \Phi(d_1)}{\partial S_0} = \phi(d_1)\frac{\partial d_1}{\partial S_0} = \frac{\phi(d_1)}{S_0 v\sqrt{T}}.$$

   Likewise, using Eq.2.29 for Δ_P,

$$\Gamma_P = \frac{\partial \Delta_P}{\partial S_0} = \frac{\partial}{\partial S_0}[\Phi(d_1) - 1] = \phi(d_1)\frac{\partial d_1}{\partial S_0} = \frac{\phi(d_1)}{S_0 v\sqrt{T}}.$$

   These both use $\frac{\partial d_1}{\partial S_0} = \frac{1}{S_0 v\sqrt{T}}$, found while computing Deltas. □

7. Derive Θ_C and Θ_P in Section 2.3.3 from Black-Scholes.

   Solution: Begin by differentiating Eq.2.25 with respect to T:

$$\Theta_C = -\frac{\partial C}{\partial T} = -S_0\Phi'(d_1)\frac{\partial d_1}{\partial T} - Kre^{-rT}\Phi(d_2) + Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial T}.$$

   Then differentiate Eqs.2.24 with respect to T:

$$\frac{\partial d_1}{\partial T} = \frac{r + \frac{v^2}{2} - \frac{1}{T}\log\frac{S_0}{K}}{2v\sqrt{T}} \quad \text{and} \quad \frac{\partial d_2}{\partial T} = \frac{r - \frac{v^2}{2} - \frac{1}{T}\log\frac{S_0}{K}}{2v\sqrt{T}}.$$

   Substitute and write $v^2/2 = v^2 - v^2/2$ to get

$$\begin{aligned}\Theta_C &= -S_0\phi(d_1)\frac{r + \frac{v^2}{2} - \frac{1}{T}\log\frac{S_0}{K}}{2v\sqrt{T}} - Kre^{-rT}\Phi(d_2) \\ &\qquad + Ke^{-rT}\phi(d_2)\frac{r - \frac{v^2}{2} - \frac{1}{T}\log\frac{S_0}{K}}{2v\sqrt{T}} \\ &= \frac{-S_0\phi(d_1)v^2}{2v\sqrt{T}} - Kre^{-rT}\Phi(d_2) \\ &\qquad - \frac{[S_0\phi(d_1) - Ke^{-rT}\phi(d_2)](r - \frac{v^2}{2} - \frac{1}{T}\log\frac{S_0}{K})}{2v\sqrt{T}},\end{aligned}$$

   after rearrangement. But $S_0\phi(d_1) - Ke^{-rT}\phi(d_2) = 0$ by Exercise 4, so the last term vanishes, leaving

$$\Theta_C = \frac{-S_0\phi(d_1)v}{2\sqrt{T}} - Kre^{-rT}\Phi(d_2),$$

   as claimed.

   To find Θ_P, rearrange the Call-Put Parity formula for European-style options as follows:

$$P(0) = C(0) - S_0 + Ke^{-rT}.$$

Then applying $-\partial/\partial T$ to both sides yields

$$\Theta_P = \Theta_C + Kre^{-rT} = \frac{-S_0\phi(d_1)v}{2\sqrt{T}} - Kre^{-rT}[\Phi(d_2) - 1].$$

But $\Phi(x) - 1 = -\Phi(-x)$ for any x by Eq.2.19, so

$$\Theta_P = \frac{-S_0\phi(d_1)v}{2\sqrt{T}} + Kre^{-r*T}\Phi(-d_2),$$

as claimed. □

8. Derive κ_C and κ_P in Section 2.3.3 from Black-Scholes.

Solution: For these "Vegas," differentiate Eqs.2.26 and 2.26 with respect to volatility v:

$$\kappa_C = \frac{\partial C}{\partial v} = S_0\Phi'(d_1)\frac{\partial d_1}{\partial v} - Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial v}.$$

Then, either by hand or using Macsyma, compute

$$\frac{\partial d_1}{\partial v} = -\frac{d_1}{v} + \sqrt{T} \qquad \text{and} \qquad \frac{\partial d_2}{\partial v} = -\frac{d_2}{v} - \sqrt{T},$$

so

$$\kappa_C = S_0\phi(d_1)[-\frac{d_1}{v} + \sqrt{T}] - Ke^{-rT}\phi(d_2)[-\frac{d_2}{v} - \sqrt{T}],$$

and after rearrangement,

$$= -[S_0\phi(d_1) - Ke^{-rT}\phi(d_2)][-\frac{d_2}{v} - \sqrt{T}] + S_0\phi(d_1)[\frac{(d_2 - d_1)}{v} + 2\sqrt{T}].$$

The first term vanishes since $S_0\phi(d_1) - Ke^{-rT}\phi(d_2) = 0$, while the second term simplifies using $d_2 - d_1 = -v\sqrt{T}$, leaving

$$\kappa_C = S_0\phi(d_1)\sqrt{T}.$$

Differentiating the Call-Put Parity formula $C(0) - P(0) = S_0 - Ke^{-rT}$ then gives

$$\kappa_P = \kappa_C = S_0\phi(d_1)\sqrt{T},$$

since the difference does not depend on v. □

9. Derive ρ_C and ρ_P in Section 2.3.3 from Black-Scholes.

Solution: Differentiate Eq.2.26 with respect to interest rate r to get

$$\rho_C = \frac{\partial C}{\partial r} = S_0\Phi'(d_1)\frac{\partial d_1}{\partial r} - K(-T)e^{-rT}\Phi(d_2) - Ke^{-rT}\Phi'(d_2)\frac{\partial d_2}{\partial r}.$$

By hand or using Macsyma, compute

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r} = \frac{\sqrt{T}}{v},$$

so

$$\rho_C = S_0\phi(d_1)\frac{\sqrt{T}}{v} + KTe^{-rT}\Phi(d_2) - Ke^{-rT}\phi(d_2)\frac{\sqrt{T}}{v},$$

which, after simplification and rearrangement, gives

$$= [S_0\phi(d_1) - Ke^{-rT}\phi(d_2)]\frac{\sqrt{T}}{v} + KTe^{-rT}\Phi(d_2).$$

The first term vanishes because $S_0\phi(d_1) - Ke^{-rT}\phi(d_2) = 0$ by Exercise 4, leaving

$$\rho_C = KTe^{-rT}\Phi(d_2).$$

Using the Call-Put Parity Formula as $P(0) = C(0) - S_0 + Ke^{-rT}$ and differentiating both sides with respect to r gives

$$\rho_P = \rho_C - 0 - KTe^{-rT} = KTe^{-rT}[\Phi(d_2) - 1] = -KTe^{-rT}\Phi(-d_2)$$

as claimed, using Eq.2.19 at the last step. □

10. (a) Verify Eq.2.34 relating Θ_C, Δ_C, and Γ_C.

(b) Find the equivalent relation for Puts and verify it.

Solution: (a) Substitute Eqs.2.28, 2.30, and 2.32 into $\Theta_C + rS_0\Delta_C + \frac{1}{2}v^2S_0^2\Gamma_C - rC(0)$ to get:

$$\frac{-S_0\phi(d_1)v}{2\sqrt{T}} - Kre^{-rT}\Phi(d_2) + rS_0\Phi(d_1) + \frac{1}{2}v^2S_0^2\frac{\phi(d_1)}{S_0v\sqrt{T}} - rC(0).$$

But this equals

$$-Kre^{-rT}\Phi(d_2) + rS_0\Phi(d_1) - rC(0) \quad = \ 0,$$

since $C(0) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$ by Eq.2.25, the Black-Scholes Call pricing formula.

(b) The equivalent relation for Puts simply replaces $C(0)$ with $P(0)$ and uses the Put Greeks:

$$\Theta_P + rS_0\Delta_P + \frac{1}{2}v^2S_0^2\Gamma_P - rP(0) = 0.$$

Check it by substituting Eqs.2.29, 2.31, and 2.33 into the left hand side to get:

$$\begin{aligned} -\frac{vS_0\phi(d_1)}{2\sqrt{T}} &+ re^{-rT}K\Phi(-d_2) \\ &+rS_0[\Phi(d_1) - 1] + \frac{1}{2}v^2S_0^2\frac{\phi(d_1)}{vS_0\sqrt{T}} - rP(0) \\ =\ & re^{-rT}K\Phi(-d_2) + rS_0[\Phi(d_1) - 1] - rP(0) \\ =\ & rKe^{-rT}\Phi(-d_2) - rS_0\Phi(-d_1) - rP(0) \qquad = 0, \end{aligned}$$

as claimed, since $Ke^{-rT}\Phi(-d_2) - S_0\Phi(d_1) = P(0)$ by Eq.2.26. The next-to-last step uses Eq.2.19 to rewrite $\Phi(d_1) - 1 = -\Phi(-d_1)$. □

11. (a) Find the coefficients p_1, p_2, p_3 that give the least-squares best fit

$$f(x) = p_1 + p_2 e^x + p_3 e^{-x}$$

to the data $\{(x, y)\} = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$.

(b) Plot f at 81 equispaced points on a graph showing the data.

Solution: (a) Use the functions $f_1(x) = 1$, $f_2(x) = e^x$, and $f_3(x) = e^{-x}$ in Eq.2.48 to find the least-squares best fit to the data. Do this by modifying the example code for general least squares regression:

```
x=[-2;-1;0;1;2]; y=[4;1;0;1;4]; % data
f1=@(v)ones(size(v)); f2=@exp; f3=@(v)exp(-v); % functions
F=@(v) [f1(v),f2(v),f3(v)];  % regression matrix function
pf=(F(x)'*F(x))\(F(x)'*y)  % least squares best fit coeffs
```

If the `optim` package is available, the last step may be done with

```
pf=LinearRegression(F(x),y)  % least squares best fit coeffs
```

Both methods give $p_1 = -1.25403$, $p_2 = 0.70066$, and $p_3 = 0.70066$.

(b) Modify the example plotting code to use 81 z values equispaced between -2 and 2:

```
z=linspace(-2,2,81)'; % must be a column vector to use with F
yf=F(z)*pf; % column vector of the best-fit function at z
plot(z,yf,"b--", x,y,"k*"); % graph of best fit versus data
title("Best fit to (x,y) by 1,exp(x),exp(-x)")
legend("best fit","(x,y)");
```

The results may be seen in Figure 1. □

12. At one instant, two days before expiry, near-the-money Call options for ABC common stock had the following prices:

Strike price	Premium	Open interest
44.00	3.80	3,260
45.00	2.77	4,499
46.00	1.77	3,862
47.00	0.78	6,271
48.00	0.18	10,156
49.00	0.03	10,619
50.00	0.01	14,219

The spot price for ABC is \$47.58. Estimate the premium for the at-the-money Call option in the following ways:

(a) Unweighted quadratic regression.

(b) Weighted quadratic regression.

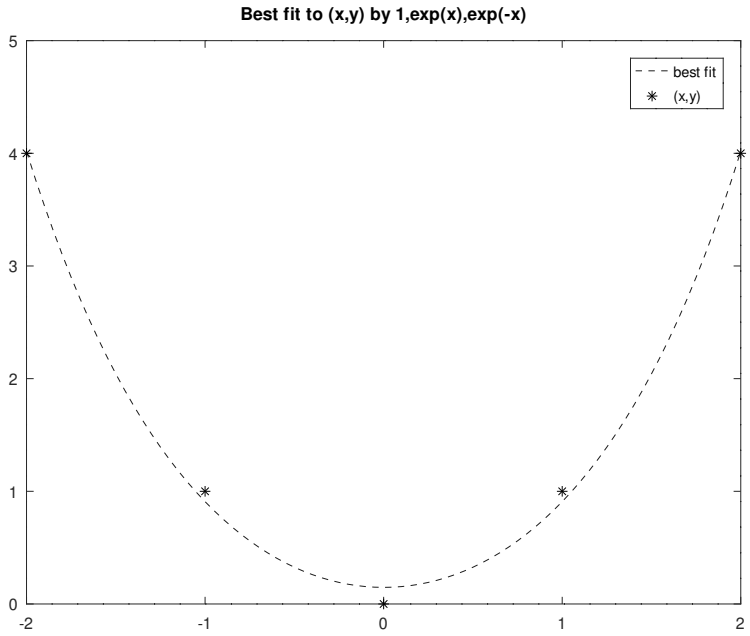


Figure 1: Graph from Exercise 11(b).

(c) Polynomial interpolation.

(d) Spline interpolation.

Solution: Use the strike prices for x, the premiums for y, and the open interest for the weights w.

(a) First, enter the data and define the quadratic functions:

```
x=[44.00; 45.00; 46.00; 47.00; 48.00; 49.00; 50.00];
y=[ 3.80;  2.77;  1.77;  0.78;  0.18;  0.03;  0.01];
f1=@(v)ones(size(v)); f2=@(v)v; f3=@(v)v.*v; % 1,v,v^2
F=@(v) [f1(v),f2(v),f3(v)];  % regression matrix function
S0=47.58;  % Spot price, to be used as the strike price
```

There are three equivalent quadratic regression methods:

```
pu=(F(x)'*F(x))\(F(x)'*y); F(S0)*pu
pua=LinearRegression(F(x),y); F(S0)*pua
polyval(polyfit(x,y,2),S0)
```

All three give the same estimate \$0.51.

(b) With the data entered in part (a), add open-interest weights as follows:

```
w=[ 3260;  4499;  3862;  6271; 10156; 10619; 14219];
W=diag(w); % weight matrix
p=(F(x)'*W*F(x))\(F(x)'*W*y);  F(S0)*p
```

Alternatively,

```
pa=LinearRegression(F(x),y,sqrt(w)); F(S0)*pa
```

Both ways give an at-the-money premium estimate \$0.47.

(c) Reusing the data entered in part (a), this is returned by

```
polyval(polyfit(x,y,length(x)-1),S0)
```

This estimate is \$0.37.

(d) With the data entered in part (a), this is returned by:

```
ppval(spline(x,y),S0)
```

A feature of Octave's `spline()` function is that it evaluates the computed spline if given a third argument:

```
spline(x,y,S0)
```

Both commands give the same estimate \$0.37.

NOTE: the market premium at this instant was \$0.38 for the Call option at strike price \$47.50. □