Math 4501/5051: Numerical Applied Mathematics

Parimutuel Midterm Examination Problems

Of the following problems, at least 5 will be on the midterm examination, possibly with minor changes.

1. (a) Convert the binary number 11 1011 1010 1001 0111 to hexadecimal.

   (b) Convert the hexadecimal numbers A12 and 1DA to decimal.

2. (a) Prove that iteration from any initial value $x_0 \in \mathbf{R}$ converges to the fixed point $x = e^{-x/10}$.

   (b) Estimate, with proof, how many iterations it would take to find the solution to within an absolute error of 5×10^{-8}, starting from $x_0 = 0$.

3. (a) Find a polynomial $p = p(x)$ of minimal degree in x such that $\cos x = p(x) + O(x^5)$ as $x \to 0$.

   (b) Find $\epsilon \in (0, 1)$ to one significant digit such that $|\cos x - p(x)| < 0.0000001 = 10^{-7}$ whenever $|x| < \epsilon$.

4. Let
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & -1 \\ 0 & 3 & 2 \end{pmatrix}.$$

   (a) Find a factorization $A = LU$, where matrix L is unit lower triangular and matrix U is upper triangular. (Hint: no interchanges are needed.)

   (b) Compute the determinant of A.

5. Let $P_0 = (0, 2)$, $P_1 = (1, 5)$, and $P_2 = (2, 6)$ be three points in the plane.

   (a) Find the Newton form of the Lagrange polynomial through P_0, P_1, P_2.

   (b) Find the Bézier curve $B = B(t) = (x(t), y(t))$, $0 \le t \le 1$, with control points P_0, P_1, P_2.

6. Find a triangular factorization $A = LU$ for these matrices:
$$(a) \quad \begin{pmatrix} 4 & 2 & 1 \\ 2 & 5 & -2 \\ 1 & -2 & 7 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 1 & -2 & 7 \\ 4 & 2 & 1 \\ 2 & 5 & -2 \end{pmatrix}$$

   (Hint: no interchanges are needed.)

7. Let T_N be the Nth Chebyshev polynomial. Use mathematical induction to prove the following:

   (a) The coefficient of x^N in $T_N(x)$ is 2^{N-1} for $N \geq 1$.

   (b) For integer $M \geq 0$, $T_{2M}(-x) = T_{2M}(x)$ for all x, namely T_N is an even function if N is an even nonnegative integer.

   (c) For integer $M \geq 0$, $T_{2M+1}(-x) = -T_{2M}(x)$ for all x, namely T_N is an odd function if N is a odd positive integer.

8. Use Aitken's method on the partial sums

$$S_n = \sum_{k=1}^{n} A_k$$

   to derive the accelerated series

$$T_n = S_n + \frac{A_{n+1}^2}{A_{n+1} - A_{n+2}}.$$

9. Find AB and BA for the following matrices:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 \\ -1 & 5 \\ 3 & -2 \end{pmatrix}$$

10. Find the parabola $y = A + Bx + Cx^2$ that passes through $(1, 6)$, $(2, 5)$, and $(3, 2)$.

11. Let

$$A = \begin{pmatrix} 5 & -1 & 1 \\ 2 & 10 & 3 \\ -2 & 1 & -9 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ -8 \\ 4 \end{pmatrix}$$

    (a) Write a recursion $x_{n+1} = F(x_n)$ to solve $Ax = b$ by Jacobi iteration from some initial guess x_0.

    (b) Prove, by applying the relevant theorem, that Jacobi iteration for $Ax = b$ will converge to a solution x from any initial x_0.

12. Find the solution A, B to the normal equations

$$\begin{aligned} A\sum_{k=1}^{N} x_k^2 + B\sum_{k=1}^{N} x_k &= \sum_{k=1}^{N} x_k y_k \\ A\sum_{k=1}^{N} x_k + B\sum_{k=1}^{N} 1 &= \sum_{k=1}^{N} x_k \end{aligned}$$

    You may assume that $N > 1$ and that there are at least two distinct values in $\{x_1, \ldots, x_N\}$.

13. Explain why evaluation $\sqrt{1+x}-1$ for $x \approx 0$ causes catastrophic cancellation. Rewrite algebraically to fix the issue.

14. Solve the lower triangular system $Ax = b$ written below:

$$\begin{aligned} 2x_1 &= 6 \\ -x_2 + 4x_2 &= 5 \\ 3x_1 - 2x_2 - x_3 &= 4 \\ x_1 - 2x_2 + 6x_3 + 3x_4 &= 2 \end{aligned}$$

    Then find $\det(A)$.

15. Let $P(x, y) = A + Bx + Cy$ be a linear polynomial. Find A, B, C such that the plane $z = P(x, y)$ passes through the points $(1, 1, 5)$, $(2, 1, 3)$, and $(1, 2, 9)$.

16. (a) Derive the cube-root iteration that solves $f(x) = x^3 - a = 0$ using Newton's method.

    (b) Find the Newton-Raphson iteration formula to solve

$$f(x) = x - \sqrt{4+x} = 0$$

    for x.

17. Suppose that $N > 1$ and $\{x_1, \ldots, x_N\}$ contains at least two distinct values. Show that

$$N \sum_{k=1}^{N} x_k^2 - \left(\sum_{k=1}^{N} x_k \right)^2 > 0$$

18. Given the Taylor polynomial expansions

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)$$

    and

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} + O(h^7),$$

    determine the order of approximation for their sum and product.

19. (a) Find a Taylor polynomial of degree $N = 5$ for $f(x) = 1/(1+x)$ expanded about $x_0 = 0$.

    (b) Find the error term $E_5(x)$ for the polynomial in part (a).

20. Write the normal equations to find the least squares curve of the form $y(x) = Ax^2 + B$ for the points $(0, 1)$, $(1, 4)$, and $(2, 12)$. It is not necessary to solve them.