Math 449: Numerical Applied Mathematics
Midterm Examination

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6 problems on 1+6 pages

No written material, no collaboration, and no electronic devices are allowed. Please write your answers in the space provided. You have 50 minutes.

Useful formulas:

- The Bézier curve through control points $P_0, \ldots, P_n$ is $B(t) = P_0 B_{0,n}(t) + \cdots + P_n B_{n,n}(t)$, $0 \le t \le 1$, where $B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$.

- **Theorem.** If there exists $K < 1$ such that $|g(x) - g(y)| \le K|x - y|$ for all x, y, then g is a contraction map and iteration $x_{n+1} = g(x_n)$ from any initial point x_0 will converge to a unique limit p satisfying

$$|x_n - p| \le |x_1 - x_0| K^n / (1 - K).$$

- **Theorem.** If f has $n+1$ continuous derivatives, then for any a, x there exists a point c betwen a and x such that

$$f(a + x) = f(a) + x f'(a)/1! + \cdots + x^n f^{(n)}(a)/n! + x^{n+1} f^{(n+1)}(c)/(n+1)!$$

- Normal equations for the least squares line $y = Ax + B$ determined by the points $\{(x_k, y_k) : k = 1, \ldots, n\}$ are

$$\begin{pmatrix} \sum x_k^2 & \sum x_k \\ \sum x_k & n \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum x_k y_k \\ \sum y_k \end{pmatrix}.$$

- Newton's form for the Lagrange polynomial through the points $\{(x_k, y_k) : k = 0, 1, \ldots, n\}$ is

$$p(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0) \cdots (x - x_{n-1}),$$

  where $c_0 = y_0, c_1 = dy_0, \ldots, c_n = d^n y_0$ is the diagonal of the divided difference table

k	x	y	dy	d^2y	$\cdots$
0	x_0	y_0			
1	x_1	y_1	$dy_0 = (y_1 - y_0)/(x_1 - x_0)$		
2	x_2	y_2	$dy_1 = (y_2 - y_1)/(x_2 - x_1)$	$d^2y_0 = (dy_1 - dy_0)/(x_2 - x_0)$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

1. (a) Prove that iteration from any initial value $x_0 \in \mathbf{R}$ converges to the fixed point $x = e^{-x^2}$.

   (b) Estimate, with proof, how many iterations it would take to find the solution to 5 decimal places of accuracy, starting from $x_0 = 0$.

   NOTE: it suffices to give a formula without actually evaluating it.

2. (a) Find a polynomial $p = p(h)$ of minimal degree in h such that $1/(1-h) = p(h) + O(h^5)$ as $h \to 0$.
   (b) Find $\epsilon > 0$ to one significant digit such that $|1/(1-h) - p(h)| < 0.00002$ whenever $|h| < \epsilon$.

3. Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 1 & 4 & -1 \\ 0 & 2 & 3 \end{pmatrix}.$$

(a) Find a factorization $A = LU$, where matrix L is unit lower triangular and matrix U is upper triangular. (Hint: no interchanges are needed.)

(b) Compute the determinant of A.

4. Let $P_0 = (0, 1)$, $P_1 = (1, 4)$, and $P_2 = (2, 3)$ be three points in the plane. Find the Newton form of the Lagrange polynomial through P_0, P_1, P_2.

5. Let $P_0 = (0, 1)$, $P_1 = (1, 4)$, and $P_2 = (2, 3)$ be three points in the plane. Find the Bézier curve $B = B(t) = (x(t), y(t))$, $0 \leq t \leq 1$, with control points P_0, P_1, P_2.

6. Let $P_0 = (0, 1)$, $P_1 = (1, 4)$, and $P_2 = (2, 3)$ be three points in the plane. Suppose $y(x) = Ax + B$ is the least squares curve for these points. Find the normal equations satisfied by A, B.