Mathematics 412: Advanced Calculus II
Problem Set 7 — due Thursday, April 18, 2002

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Suppose that $f_i : \mathbf{R} \to \mathbf{R}$ is defined and bounded on the compact interval $[a_i, b_i] \subset \mathbf{R}$. If $f_i \in R([a_i, b_i])$ for $i = 1, \ldots, n$, prove that

$$\int_Q f_1(x_1) \cdots f_n(x_n)\, d(x_1, \ldots, x_n) = \left(\int_{a_1}^{b_1} f_1(x_1)\, dx_1 \right) \cdots \left(\int_{a_n}^{b_n} f_n(x_n)\, dx_n \right),$$

where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbf{R}^n$.

Problem 2: Let $Q = [0, 1] \times [0, 1]$ and $f(x, y) = x^2 + y^2$. Compute $\int_Q f(x, y)\, d(x, y)$.

Problem 3: Let $S = \{(x, y) \in \mathbf{R}^2 : |x| + |y| \leq 1, x \geq y \geq 0\}$ and put $f(x, y) = x^2 + y^2$. Compute $\int_S f(x, y)\, d(x, y)$.

Problem 4: Let $Q = [0, 1] \times [0, 1] \subset \mathbf{R}^2$ and Define $f : Q \to \mathbf{R}$ by

$$f(x, y) = \begin{cases} 0, & \text{if at least one of } x, y \text{ is irrational;} \\ 1/n, & \text{if } y \text{ is rational and } x = m/n, \end{cases}$$

where m, n are relatively prime nonnegative integers expressing the rational number x in lowest terms. Prove the following facts about Riemann integrals:

(a) $\int_0^1 f(x, y)\, dx = 0$ exists,
(b) $\int_0^1 \left[\int_0^1 f(x, y)\, dx \right] dy = 0$ exists,
(c) $\int_Q f(x, y)\, d(x, y) = 0$ exists, but
(d) $\int_0^1 f(x, y)\, dy$ does not exist for any rational $x \in [0, 1]$.

Problem 5: Suppose that $S \subset \mathbf{R}^n$ is a bounded set having finitely many accumulation points. Prove that $c(S) = 0$.

Problem 6: Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function defined on $[a, b] \subset \mathbf{R}$, and let $S = \{(x, y) \in \mathbf{R}^2 : x \in [a, b], y = f(x)\}$ be the graph of f. Prove that the two-dimensional Jordan content $c(S)$ is zero.

Problem 7: Let S be a bounded line segment in $\mathbf{R}^n$, $n \geq 2$. Prove that S has n-dimensional Jordan content zero.

Problem 8: Define

$$f(x,y) = \begin{cases} e^{-xy} \sin x \sin y, & \text{if } x \geq 0 \text{ and } y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Prove that both iterated integrals exist, with

$$\int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)\, dx \right] dy = \int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)\, dy \right] dx,$$

but that the double integral of f over $\mathbf{R}^2$ does not exist. Why does this not contradict the Tonelli-Hobson test (Th.15.8, p.415)?

Problem 9: Prove that $\int_{\mathbf{R}^2} e^{-x^2-y^2}\, d(x,y) = \pi$ by transforming to polar coordinates. Use this to prove that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

Problem 10: Let V_n denote the n-measure of the n-ball $B(0;1)$ of radius 1. Prove that $V_n = \pi^{n/2}/\Gamma(\frac{1}{2}n + 1)$, where

$$\Gamma(s+1) \stackrel{\text{def}}{=} \int_0^{\infty} t^s e^{-t}\, dt$$

exists as an improper Riemann integral for all $s > -1$.