Mathematics 412: Advanced Calculus II
Problem Set 6 — due Tuesday, April 2, 2002

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

For problems 1, 2, and 3, let $\mathbf{f} = (f_1, f_2, f_3)$ be the $\mathbf{R}^3$-valued function defined by

$$f_k(x_1, x_2, x_3) = \frac{x_k}{1 + x_1 + x_2 + x_3}, \qquad k = 1, 2, 3,$$

for all $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$ satisfying $x_1 + x_2 + x_3 \neq -1$.

Problem 1: Compute the Jacobian determinant $J_{\mathbf{f}}(\mathbf{x})$ at all $\mathbf{x}$ in the domain of $\mathbf{f}$.

Problem 2: Show that $\mathbf{f}$ is one-to-one.

Problem 3: Find the inverse function $\mathbf{f}^{-1}$.

Problem 4: Let $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. Compute $\dfrac{\partial(x, y)}{\partial(r, \theta)}$.

Problem 5: Let $x(r, \theta, \phi) = r\cos\theta\sin\phi$, $y(r, \theta, \phi) = r\sin\theta\sin\phi$, and $z(r, \theta, \phi) = r\cos\phi$. Compute

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}.$$

Problem 6: Find and classify the extreme values of the function $f(x, y) = x^2 + 2y^2 - x - y + xy$.

Problem 7: (a) Find the maximum of $(x_1 x_2 \cdots x_n)^2$ under the restriction $x_1^2 + \cdots + x_n^2 = 1$. (b) Use the result to derive the following inequality, valid for all positive real numbers $a_1, \ldots, a_n$:

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Problem 8: Fix arbitrary $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n$, nonzero $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbf{R}^n$, and $c \in \mathbf{R}$. Find the shortest distance from the point $\mathbf{a}$ to the hyperplane $P = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{b} \cdot \mathbf{x} = c\}$.

Problem 9: Let $X = [x_{ij}]$ be an $n \times n$ matrix, with $n \geq 2$. For each $1 \leq i, j \leq n$, define the *ij-minor matrix* to be the $(n-1) \times (n-1)$ matrix M^{ij} consisting of X without row i or column j. Likewise, define the *cofactor of* x_{ij} to be $\tilde{x}_{ij} \stackrel{\text{def}}{=} (-1)^{i+j} \det M^{ij}$, and the *dual matrix* $\tilde{X} = [\tilde{x}_{ij}]$ to be the matrix of cofactors. Prove that $\nabla \det X = \tilde{X}$. (Hint: use the matrix identity $\tilde{X} X^T = (\det X) I$.)

Problem 10: Let $X = [x_{ij}]$ be an $n \times n$ matrix, and let $X_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n$ be its i^{th} row vector. Prove *Hadamard's inequality*, $|\det X| \leq \|X_1\| \cdots \|X_n\|$. (Hint: treat $|\det X|$ as a function of n^2 variables with the n constraints $\|X_i\| = d_i$, for $i = 1, \ldots, n$, and use Lagrange's method with Problem 9.)