Mathematics 412: Advanced Calculus II
Problem Set 5 — due Tuesday, March 19, 2002

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Suppose $f : S \to \mathbf{R}$ for an open $S \subset \mathbf{R}^n$ and f has finite partial derivatives $D_1 f, \ldots, D_n f$ on S. Prove that if f has a local maximum or a local minimum at a point $\mathbf{c} \in S$, then $D_k f(\mathbf{c}) = 0$ for $k = 1, 2, \ldots, n$.

Problem 2: Let $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ be defined by $\mathbf{f}(\theta, \phi) = (\sin\theta \sin\phi, \cos\theta \sin\phi, \cos\phi)$. Compute the Jacobian matrix of $\mathbf{f}$.

Problem 3: Compute the gradient vector $\nabla f(x, y)$ as a function of (x, y), and state where it exists, for the following functions:

(a) $f(x, y) = x^2 y^2 \log(x^2 + y^2)$ if $(x, y) \neq (0, 0)$; $\qquad f(0, 0) = 0$.

(b) $f(x, y) = xy \sin\left(\frac{1}{x^2+y^2}\right)$ if $(x, y) \neq (0, 0)$; $\qquad f(0, 0) = 0$.

Problem 4: Suppose f is a real-valued function defined and differentiable on $\mathbf{R}^2$ and define $F(r, \theta) = f(r\cos\theta, r\sin\theta)$. Verify the formula $\|\nabla f(r\cos\theta, r\sin\theta)\|^2 = [D_1 F(r, \theta)]^2 + \frac{1}{r^2}[D_2 F(r, \theta)]^2$.

Problem 5: If f and g have gradients $\nabla f(\mathbf{c})$ and $\nabla g(\mathbf{c})$ at $\mathbf{c} \in \mathbf{R}^n$, show that $h \stackrel{\text{def}}{=} fg$ has a gradient at $\mathbf{c}$ and $\nabla h(\mathbf{c}) = f(\mathbf{c})\nabla g(\mathbf{c}) + g(\mathbf{c})\nabla f(\mathbf{c})$.

Problem 6: Suppose that f is a real-valued function defined on an open ball $S = B(0; r) \subset \mathbf{R}^n$. We say that f is *homogeneous of degree p over S* if $f(\lambda\mathbf{x}) = \lambda^p f(\mathbf{x})$ for all $x \in S$ and all $\lambda \in [-1, 1]$. Prove that, if such an f is differentiable at x, then $\mathbf{x} \cdot \nabla f(\mathbf{x}) = p f(\mathbf{x})$.

Problem 7: Let f be a real-valued function differentiable in an n-ball $B = B(\mathbf{c})$. Suppose that $\mathbf{x} \in B$, $\mathbf{x} = (x_1, \ldots, x_n)$; prove that there exist numbers $\xi_1, \ldots, \xi_n$ with ξ_k strictly between c_k and x_k for $k = 1, \ldots, n$, such that $f(\mathbf{x}) - f(\mathbf{c}) = (x_1 - c_1)D_1 f(\xi_1, x_2, \ldots, x_n) + (x_2 - c_2)D_2 f(c_1, \xi_2, x_3, \ldots, x_n) + \cdots + (x_n - c_n)D_n f(c_1, c_2, \ldots, c_{n-1}, \xi_n)$.

Problem 8: Find the Taylor polynomial of degree 2 for the function $f(x, y) = \cos(x + y)e^{-xy}$ at the point $(0, 0)$.

Problem 9: Suppose that $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable at each point $\mathbf{x} \in \mathbf{R}^n$ and has a differentiable inverse function $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^n$. Prove that the Jacobian matrix $D\mathbf{f}(\mathbf{x})$ is nonsingular at each $\mathbf{x} \in \mathbf{R}^n$ and that $D\mathbf{g}(\mathbf{x}) = [D\mathbf{f}(\mathbf{g}(\mathbf{x}))]^{-1}$.

Problem 10: Prove that there is no function $f : \mathbf{R}^n \to \mathbf{R}$ differentiable at the point $\mathbf{c} \in \mathbf{R}^n$ and satisfying $f'(\mathbf{c}; \mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{R}^n$. Find a function $f : \mathbf{R}^n \to \mathbf{R}$ differentiable at each $\mathbf{c} \in \mathbf{R}^n$ and a fixed $\mathbf{x} \in \mathbf{R}^n$ such that $f'(\mathbf{c}; \mathbf{x}) > 0$ for all $\mathbf{c} \in \mathbf{R}^n$.