Mathematics 412: Advanced Calculus II
Problem Set 2 — due Tuesday, February 5, 2002

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Suppose that $f_n \to f$ and $g_n \to g$ both converge uniformly on S. Prove that $f_n - g_n$ converges uniformly to $f - g$ on S.

Problem 2: Define $h_n(x) = f_n(x)g_n(x)$, where $f_n(x) = (1 + \frac{1}{n})x$ and

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ b + \frac{1}{n}, & \text{if } x = \frac{a}{b} \text{ in lowest terms, with } b > 0 \text{ and } a \neq 0. \end{cases}$$

(a) Prove that $\{f_n\}$ and $\{g_n\}$ converge uniformly on every bounded interval.
(b) Prove that $\{h_n\}$ does **not** converge uniformly on any bounded interval.

Problem 3: Find a set of hypotheses such that, if $f_n \to f$ and $g_n \to g$ both converge uniformly on S, then $f_n g_n \to fg$ will converge uniformly on S. Do the same for f_n/g_n.

Problem 4: Assume that
(1) $f_n \to f$ uniformly on S;
(2) Each f_n is continuous on S;
(3) The sequence $\{x_n\} \subset S$ converges to $a \in S$.
Prove that $f_n(x_n) \to f(a)$ as $n \to \infty$.

Problem 5: Find counterexamples to show that each assumption 1, 2, and 3 is necessary in Problem 4.

Problem 6: Let $f_n(x)$ be the real-valued function defined on $[0, 1]$ by the formula

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 2^{-n}; \\ 2^{n/2}, & \text{if } 2^{-n} \leq x \leq 2^{1-n}; \\ 0, & \text{if } 2^{1-n} < x \leq 1, \end{cases}$$

for $n = 1, 2, \ldots$. Prove that $\{f_n\}$ converges pointwise to 0 on $[0, 1]$, but l.i.m.$_{n\to\infty} f_n \neq 0$.

Problem 7: Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, with $f(0) \stackrel{\text{def}}{=} 0$. Prove that $f^{(n)}(0)$ exists and equals 0 for each $n = 0, 1, 2, \ldots$.

Problem 8: Suppose that $g(x)$ is continuous on $[0, 1]$ and $g(1) = 0$. Define $f_n(x) = g(x)x^n$. Prove that $f_n \to 0$ converges uniformly on $[0, 1]$ as $n \to \infty$.

Problem 9: Suppose $f(x)$ is represented by the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 = a_1 = 3$ while $a_n = 1$ for $n \geq 2$. What is the power series for $1/f(x)$, and its radius of convergence?

Problem 10: Suppose that $\{a_n\}$ is a sequence of complex numbers and $\sum_{n=0}^{\infty} a_n$ converges. Prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly for $0 \leq x \leq 1$. Deduce Abel's theorem as a corollary.