Mathematics 412: Advanced Calculus II
Problem Set 1 — due Tuesday, January 22, 2002

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Let $a_n = \frac{n^n}{n!}$ for $n > 0$. Prove that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e$.

Problem 2: Suppose that $\{a_n\}$ is a bounded sequence of real numbers with the property that $\liminf_{n\to\infty} a_n \geq \limsup_{n\to\infty} a_n$. Prove that $\lim_{n\to\infty} a_n$ exists.

Problem 3: Determine with proof whether the following series converges:

$$\sum_{n=1}^{\infty} \left(\sqrt{1+n^6} - n^3 \right)$$

Problem 4: Determine with proof whether the following series converges:

$$\sum_{n=1}^{\infty} \left(\sqrt{1+n} - \sqrt{n} \right)$$

Problem 5: Determine with proof whether the following series converges:

$$\sum_{n=2}^{\infty} (\log n)^{-\log n}$$

Problem 6: Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Define $\{b_n\}$ by

$$b_n = \begin{cases} a_n, & \text{if } n \text{ is even,} \\ -a_n^2, & \text{if } n \text{ is odd.} \end{cases}$$

Prove that $\sum_{n=1}^{\infty} b_n$ converges absolutely.

Problem 7: Find a double sequence $\{a_{n,m}\}$ such that $\lim_{n\to\infty} a_{n,m} = 0$ for all fixed m and $\lim_{m\to\infty} a_{n,m} = 0$ for all fixed n, but $\lim_{n,m\to\infty} a_{n,m}$ does not exist.

Problem 8: Prove that if a double series converges absolutely, then it converges.

Problem 9: Find the Cesàro sum of the complex-valued series $\sum_{n=0}^{\infty} i^n$, where $i^2 = -1$.

Problem 10: Prove that $\prod_{n=2}^{\infty}(1 - n^{-2})$ converges and evaluate it.