Mathematics 411: Advanced Calculus I
Problem Set 7 — Due Tuesday, December 4, 2001

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Suppose that every monotonic f belongs to $R(\alpha)$ on $[a,b]$ and satisfies $\int_a^b f\,d\alpha = 0$. Prove that α must be a constant function on $[a,b]$.

Problem 2: Define $\phi(x) = x - \lfloor x \rfloor - \frac{1}{2}$ for noninteger x, with $\phi(x) = 0$ for integers x, and define $\psi(x) = \int_0^x \phi(t)\,dt$. If f is a function for which f'' is continuous on $[1,n]$, prove that Euler's summation formula implies that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx - \int_1^n \psi(x) f''(x)\,dx + \frac{f(1)+f(n)}{2}.$$

Problem 3: Using the formula of Problem 2 with $f(x) = \log x$, prove that

$$\log n! = (n+\frac{1}{2})\log n - n + 1 + \int_1^n \frac{\psi(x)}{x^2}\,dx.$$

Problem 4: Give an example of a bounded function f and an increasing function α defined on $[a,b]$ such that $|f| \in R(\alpha)$ but $f \notin R(\alpha)$.

Problem 5: Assume that α has bounded variation on $[a,b]$ and $f \in R(\alpha)$ on $[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$, where $a < x \le b$, and put $V(a) = 0$ as usual. Show that

$$\left| \int_a^b f\,d\alpha \right| \le \int_a^b |f|\,dV \le MV(b),$$

where $M = \sup\{|f(t)| : a \le t \le b\}$.

Problem 6: Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a vector-valued function with continuous derivative $\mathbf{f}'$ on $[a,b]$. Prove that the curve described by $\mathbf{f}$ has length

$$\Lambda_{\mathbf{f}}(a,b) = \int_a^b \|\mathbf{f}'(t)\|\,dt.$$

Problem 7: Let f be a positive continuous function in $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n \, dx \right)^{1/n} = M.$$

Problem 8: Assume that f has a decreasing derivative which satisfied $f'(x) \geq m > 0$ for all $x \in [a, b]$. Prove that

$$\left| \int_a^b \cos f(x) \, dx \right| \leq \frac{2}{m}.$$

(Hint: Multiply and divide the integrand by $f'(x)$ and use Theorem 7.37(ii)).

Problem 9: Prove that the following function is Riemann integrable on $[0, 1]$:

$$f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \in (0, 1) \text{ is irrational;} \\ 1/n, & \text{if } x \in (0, 1] \text{ is rational, with } x = m/n \text{ in lowest terms.} \end{cases}$$

(Hint: compute the oscillation $\omega_f(x)$ of f at each $x \in [0, 1]$.)

Problem 10: Define

$$g(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } 0 < x \leq 1. \end{cases}$$

(a) Prove that g is Riemann integrable on $[0, 1]$.

(b) Let $f = f(x)$ be as in Problem 9. Prove that $g \circ f$ is not Riemann integrable on $[0, 1]$, despite both $f \in R$ and $g \in R$ on $[0, 1]$.