**Mathematics 411: Advanced Calculus I**
**Problem Set 5 — Due Thursday, November 1, 2001**

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Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

**Problem 1:** Prove that a metric space $S$ is disconnected if and only if there is a subset $A \subset S$ which is neither empty nor all of $S$, but which is both open and closed in $S$.

**Problem 2:** A set $S \subset \mathbf{R}^n$ is called *starlike* if there is some *base point* $x \in S$ such that for every point $y \in S$, the line between $x$ and $y$ is contained in $S$.

(a) Prove that every convex subset of $\mathbf{R}^n$ is starlike.

(b) Prove that every starlike subset of $\mathbf{R}^n$ is connected.

**Problem 3:** Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and one-to-one on a compact interval $[a, b]$, then $f$ must be strictly monotonic on $[a, b]$.

**Problem 4:** A function $f : \mathbf{R} \to \mathbf{R}$ is said to satisfy a *Lipschitz condition of order* $\alpha$ at a point $c$ in its domain if $(\exists M > 0)(\exists r > 0)(\forall x \in B(c; r), x \neq c)|f(x) - f(c)| < M|x - c|^\alpha$.

(a) Prove that if $f$ satisfies a Lipschitz condition of order $\alpha > 0$ at $c$, then $f$ is continuous at $c$.

(b) Prove that if $f$ satisfies a Lipschitz condition of order $\alpha > 1$ at $c$, then $f$ is differentiable at $c$.

(c) Find a function $f$ satisfying a Lipschitz condition of order $\alpha = 1$ at $c$ but which is not differentiable at $c$.

**Problem 5:** Suppose that $f$ is defined on $(0, 1]$ and has a bounded derivative in $(0, 1)$ (*i.e.*, there is a finite $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (0, 1)$). Put $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} a_n$ exists. (Hint: use the Cauchy criterion.)

**Problem 6:** Let $f$ be continuous on $[0, 1]$ with $f(0) = 0$ and $f'(x)$ finite at each $x \in (0, 1)$. Suppose $f'(x)$ is an increasing function on $(0, 1)$. Prove that $g(x) \stackrel{\text{def}}{=} f(x)/x$ is an increasing function on $(0, 1)$.

**Problem 7:** Prove that if $f$ has a finite third derivative $f'''$ in $(a, b)$, and $f(a) = f(b) = f'(a) = f'(b) = 0$, then there must be some point $c \in (a, b)$ for which $f'''(c) = 0$.

**Problem 8:** Suppose that the vector-valued function $\mathbf{f}$ is differentiable at each point $t \in (a, b)$, and that $\|\mathbf{f}\|$ is constant on $(a, b)$. Prove that $\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0$ for all $t \in (a, b)$.

**Problem 9:** Define a real-valued function $f$ of two real variables as follows:

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq 0; \qquad f(0, 0) = 0.$$

(a) Prove that the partial derivatives $D_1 f(x, y)$ and $D_2 f(x, y)$ exist for every $(x, y) \in \mathbf{R}^2$ and find explicit formulas for them.

(b) Show that $f$ is not continuous at $(0, 0)$.

**Problem 10:** Let $S$ be an open set in $\mathbf{C}$ and let $S^*$ be the set of complex conjugates of points of $S$: $S^* \stackrel{\text{def}}{=} \{\bar{z} : z \in S\}$. If $f$ is defined on $S$, define $g$ on $S^*$ by the formula $g(z) = \overline{f(\bar{z})}$. Prove that if $f$ is differentiable at $c \in \text{int}S$, then $g$ is differentiable at $\bar{c}$.