Mathematics 411: Advanced Calculus I
Problem Set 3 — due Thursday, October 4, 2001

Prof. M. V. Wickerhauser

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions.

Problem 1: Prove that every closed subset of $\mathbf{R}$ is the intersection of a countable collection of open sets.

For problems 2–3, a set $S \subset \mathbf{R}^n$ is called *convex* if for every pair of points $\mathbf{x}, \mathbf{y} \in S$ and every real number θ satisfying $0 < \theta < 1$ we have $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in S$.

Problem 2: Prove that (a) an n-ball is convex; and (b) an n-dimensional open interval is convex.

Problem 3: Prove that the intersection of any collection of convex sets is convex.

Problem 4: Prove that the collection of isolated points of a set $S \subset \mathbf{R}^n$ must be countable.

Problem 5: (a) Give an example of a closed subset $U \subset \mathbf{R}$ which is not bounded and an infinite open cover F of U which has no finite subcover. (b) Give an example of a bounded subset $B \subset \mathbf{R}$ which is not closed and an infinite open cover F of B which has no finite subcover.

For Problems 6 and 7, let S be a subset of $\mathbf{R}^n$ and define a *condensation point of* S to be any point $\mathbf{x} \in \mathbf{R}^n$ such that every n-ball centered at $\mathbf{x}$ contains uncountably many points of S.

Problem 6: Prove that every uncountable subset $S \subset \mathbf{R}^n$ contains a condensation point of S. (Hint: use the fact that the countable union of countable sets is countable).

Problem 7: Assume that S is an uncountable subset of $\mathbf{R}^n$. Let T be the collection of all condensation points of S. Prove the following: (a) $S - T$ is countable; (b) $S \cap T$ is uncountable; (c) T is closed; (d) T contains no isolated points.

For Problems 8–10, let $\|\mathbf{x}\|$ be the usual norm of $\mathbf{x} \in \mathbf{R}^n$, and define $\|\mathbf{x}\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^n |x_i|$ and $\|\mathbf{x}\|_\infty \stackrel{\text{def}}{=} \max_{i=1,2,\ldots,n} |x_i|$ for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$.

Problem 8: Show that $\|\cdot\|_1$ satisfies the norm axioms:
(i) $\|\mathbf{x}\|_1 \geq 0$, with equality if and only if $\mathbf{x} = 0$.
(ii) $\|a\mathbf{x}\|_1 = |a|\,\|\mathbf{x}\|_1$, for any scalar a and vector $\mathbf{x}$.
(iii) $\|\mathbf{x} + \mathbf{y}\|_1 \leq \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$.

Problem 9: Show that $\|\cdot\|_\infty$ satisfies the three norm axioms of Problem 8.

Problem 10: Find four positive constants A, B, C, D such that for every $\mathbf{x}$ in $\mathbf{R}^n$ we have

$$A\|\mathbf{x}\| \leq \|\mathbf{x}\|_1 \leq B\|\mathbf{x}\| \qquad \text{and} \qquad C\|\mathbf{x}\| \leq \|\mathbf{x}\|_\infty \leq D\|\mathbf{x}\|.$$

For full credit, find the largest A and C and the smallest B and D for which these inequalities hold, and prove that no "better" numbers exist. (Hint: find examples in $\mathbf{R}^1$ or $\mathbf{R}^2$ which require the "best" constants A, B, C, D.)