Ma 4102: Introduction to Lebesgue Integration
Homework Assignment 6

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Read Chapter 9 of our textbook.
Upload your complete solutions using GradeScope. **Late homework will not be accepted.**
Note: Many of the exercises in section 42 of Chapter 9, in addition to those assigned, are worthy of your efforts.

1. (Ex.42.1,p.144) Show that if the Fourier series of $f \in \mathcal{L}^2([-\pi,\pi])$ converges pointwise a.e., then it converges pointwise a.e. to f.

   [HINT: Show, as in the proof of Th.33.11, p.106, that a subsequence of the Fourier partial sums $\{s_n\}$ converges pointwise a.e. to f.]

2. (Ex.42.2,p.144) Consider the Fourier series of Example 35.3 on page 116, namely for the function $\chi_{[0,\pi]}$.

   (a) Use it and an appropriate theorem to prove that

   $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

   [HINT: put $x = \pi/2$.]

   (b) Let $x = \pi/4$ and obtain another series representation for $\pi/4$.

   (c) Which series, (a) or (b), needs fewer terms n to reach the accuracy $|s_n - \pi/4| < 10^{-7}$?

3. (Ex.42.4,p.144) Suppose that $f : [-\pi,\pi] \to \mathbf{R}$ is 2π-periodic, continuous on $[-\pi,\pi]$, and differentiable at $x_0 \in [-\pi,\pi]$. For $t \in (-\pi,\pi)$, $t \neq 0$, define

   $$h(t) \stackrel{\text{def}}{=} \frac{f(x_0+t) + f(x_0-t) - 2f(x_0)}{2\sin(t/2)}.$$

   Show that there is a number $y_0 \in \mathbf{R}$ such that by defining $h(0) \stackrel{\text{def}}{=} y_0$, the resulting function is continuous on $(-\pi,\pi)$.

4. (Ex.42.6,p.144) Show that if $f \in \mathcal{L}^2([-\pi,\pi])$, then

   $$\lim_{n\to\infty} \int_0^\pi f(t) \sin(n + \frac{1}{2})t \, dt = 0.$$

   [HINT: Th.36.4, p.118, and angle addition formulas.]

5. (Ex.42.8,p.144) Given $f \in \mathcal{L}([-\pi,\pi])$ and $\epsilon > 0$, show that there is a simple function g on $[-\pi,\pi]$ such that

   $$\int_{[-\pi,\pi]} |f - g| \, d\mu < \epsilon.$$

   [HINT: use f^+ and f^-.]

6. (Ex.42.9,p.144) Prove Theorem 41.7.

7. (Ex.42.11,p.144)

   (a) For $x \in [0, \pi)$, find the Fourier series (namely, the coefficients $\{a_n, b_n\}$) for the function $\chi_{[0,x)}$.

   (b) Use part (a) and appropriate theorems to obtain the identity

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

8. (Ex.42.12,p.145) Let $\{a_0; a_n, b_n, n = 1, 2, 3, \dots\} \subset \mathbf{R}$ be the coefficients giving a sequence of trigonometric series

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \qquad n = 0, 1, 2, \dots.$$

   (a) Show that if $\{s_n(c)\}$ converges absolutely, then $\{s_n(-c)\}$ also converges absolutely.

   (b) Show that if $\{s_n(c_1)\}$ and $\{s_n(c_2)\}$ both converge absolutely, then $\{s_n(c_1+c_2)\}$ converges absolutely.

   (c) Show that if $\{s_n(x)\}$ converges absolutely for all x in some nonempty open interval, then it converges absolutely at all $x \in [-\pi, \pi]$.

9. (Extra A) Using Euler's formula $e^{ix} = \cos x + i \sin x$, it is possible to write a trigonometric polynomial in terms of complex exponential functions:

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=-n}^{n} c_k e^{ikx}.$$

   (a) Find the formulas relating (c_k, c_{-k}) and (a_k, b_k), for $k = 0, 1, 2, \dots$.

   (b) Show that $\{e^{ikx} : k \in \mathbf{Z}\}$ is an orthogonal set of functions in $\mathcal{L}^2([-\pi, \pi])$ with respect to the *Hermitean inner product*

$$f \cdot g \stackrel{\text{def}}{=} \int_{[-\pi,\pi]} f \bar{g} \, d\mu,$$

   where $\bar{g}$ is the complex conjugate of g, so for example $\overline{e^{ix}} = e^{-ix}$ for all real x.

10. (Extra B) For integer $N > 0$, let $A_N \stackrel{\text{def}}{=} [-N, N] \subset \mathbf{R}$. Say that $f \in \mathcal{L}(\mathbf{R})$ if

    - $f \in \mathcal{L}(A_N)$ for every $N = 1, 2, 3, \dots$, and
    - $\|f\| \stackrel{\text{def}}{=} \lim_{N\to\infty} \int_{A_N} |f| \, d\mu$ exists and is finite.

    (a) Prove that if $f \in \mathcal{L}(\mathbf{R})$ and g is a bounded measurable function on $\mathbf{R}$, then $gf \in \mathcal{L}(\mathbf{R})$.

    (b) Prove that if $f, g \in \mathcal{L}(\mathbf{R})$ then $f + g \in \mathcal{L}(\mathbf{R})$.

    (c) For $f(x) = e^{-\pi x^2}$, compute

$$\int_{\mathbf{R}} f \, d\mu = \lim_{N\to\infty} \int_{A_N} f \, d\mu$$

    [HINT: With justification, use results from Calculus.]