Ma 404: Numerical Methods
Midterm Examination

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You may use a calculator and the textbook. Please write your answers in the bluebook.

Problem 1. Find the sum of the series

$$\sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k$$

Solution: Use

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad |r| < 1$$

with $r = -\frac{1}{4}$ to get 4/5.

Problem 2. Suppose $f(h) = 1 - h^2 + O(h^3)$, while $g(h) = 1 + h^2 + O(h^4)$ as $h \to 0$. Give the best possible formula for $f(h)g(h)$, and determine its order of approximation as $h \to 0$.

Solution: Multiply $f(h)g(h) = 1 - h^4 + O(h^3)$, but $h^4 + O(h^3) = O(h^3)$ as $h \to 0$, so the best possible formula is $f(h)g(h) = 1 + O(h^3)$.

Problem 3. Solve the equation $x = g(x)$ for $g(x) = \sqrt{4+x}$ to 5 significant digits.

Solution: Answer is 2.56155, obtained after 10 iterations from an initial value of 2.

Problem 4. Find the Newton-Raphson iteration formula for $f(x) = x - \sqrt{4+x}$.

Solution: $f'(x) = 1 - \frac{1}{2\sqrt{4+x}}$, so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \sqrt{4+x_n}}{1 - \frac{1}{2\sqrt{4+x_n}}} = \frac{8+x_n}{2\sqrt{4+x_n}-1}.$$

Problem 5. Find the determinant $\det A$ and the inverse matrix A^{-1} for

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}.$$

Solution:

$$A = \frac{1}{13}\begin{pmatrix} 5 & -1 \\ -2 & 3 \end{pmatrix}; \qquad \det A = 13.$$

Problem 6. Let $\mathbf{x} = (1, -1, 2)$ and $\mathbf{y} = (3, 0, 4)$ be two vectors in $\mathbf{R}^3$. Compute $\|\mathbf{x}\|$, $\|\mathbf{y}\|$, $\mathbf{x} \cdot \mathbf{y}$, and the angle between $\mathbf{x}$ and $\mathbf{y}$.

Solution: $\|\mathbf{x}\| = \sqrt{6}$, $\|\mathbf{y}\| = 5$, $\mathbf{x} \cdot \mathbf{y} = 11$, and the angle between $\mathbf{x}$ and $\mathbf{y}$ is $\cos^{-1} \frac{11}{5\sqrt{6}} \approx 0.45526 \approx 26°$.

Problem 7. Find a factorization $A = LU$, where L is unit lower triangular and U is upper triangular, for

$$A = \begin{pmatrix} 9 & 3 & 0 \\ 3 & 8 & -1 \\ 0 & 1 & 7 \end{pmatrix}.$$

(Hint: no interchanges are needed.)

Solution:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & \frac{1}{7} & 1 \end{pmatrix}; U = \begin{pmatrix} 9 & 3 & 0 \\ 0 & 7 & -1 \\ 0 & 0 & \frac{50}{7} \end{pmatrix}.$$

Problem 8. Find the quadratic Lagrange polynomial that interpolates $(0, 1)$, $(1, 2)$, and $(3, 10)$. Evaluate it at $x = 2$.

Solution: $p(x) = x^2 + 1$, so $p(2) = 5$.

Problem 9. Find the least squares curve of the form $y(x) = \frac{A}{x} + B$ for the points $(1, 3)$, $(2, 2)$, and $(5, 1)$.

Solution: Linearize by $X_k \leftarrow (1/x_k)$, $Y_k \leftarrow y_k$ and use the normal equations:

$$\begin{pmatrix} \sum_k X_k^2 & \sum_k X_k \\ \sum_k X_k & N \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \sum_k X_k Y_k \\ \sum_k Y_k \end{pmatrix}.$$

But $N = 3$, $\sum_k X_k = 1 + 0.5 + 0.2 = 1.7$, $\sum_k X_k^2 = 1 + 0.25 + 0.04 = 1.29$, $\sum_k Y_k = 3 + 2 + 1 = 6$, and $\sum_k X_k Y_k = 3 + 1 + 0.2 = 4.2$. Hence the normal equations are

$$\begin{pmatrix} 1.29 & 1.7 \\ 1.7 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4.2 \\ 6 \end{pmatrix} \implies \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{0.98} \begin{pmatrix} 3 & -1.7 \\ -1.7 & 1.29 \end{pmatrix} \begin{pmatrix} 4.2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2.449 \\ 0.612 \end{pmatrix}.$$

Hence the least-squares curve is $y = \frac{2.449}{x} + 0.612$.