Ma 3520: Differential Equations and Dynamical Systems
Homework Assignment 6

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Read Chapters 11 and 12 of the textbook, "Nonlinear Dynamics and Chaos," 3rd ed., by Steven H. Strogatz. Upload your complete solutions using GradeScope. **Late homework will not be accepted.**

Do the following exercises:

1. (Ex.11.1.6,p.453) The *decimal shift map* is $x_{n+1} = 10x_n \pmod 1$. Initial values x_0 are decimal expansions of real numbers in $[0,1]$.

   (a) Show that the map has countably many periodic orbits.

   (b) Show that the map has uncountably many aperiodic orbits.

   (c) An *eventually fixed point* of an iteration is an initial value x_0 such that, for some $N \geq 0$, we have $x_n = x_N$ for all $n \geq N$. Is the number of eventually fixed points for the decimal shift map countable or uncountable?

2. (Ex.11.2.3,p.454) The *outer measure* of a subset $S \subset \mathbf{R}$ is defined by

$$\mu(S) \stackrel{\text{def}}{=} \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : S \subset \bigcup_{i=1}^{\infty}(a_i, b_i)\},$$

   namely the greatest lower bound of the sum of the lengths of a countable cover of S by open intervals $\{(a_i, b_i) : i = 1, 2, \dots\}$.

   Show that any countable subset $S = \{x_1, x_2, \dots\}$ has outer measure zero.

3. (Ex.11.3.1,p.455) The *middle halves Cantor set* H is a fractal constructed in stages from the interval $S_0 = [0,1]$ by removing the middle half of each subinterval S_n to get S_{n+1}, then letting

$$H \stackrel{\text{def}}{=} \bigcap_{n=0}^{\infty} S_n$$

   (a) Find the similarity dimension of H.

   (b) Find the box dimension of H.

   (c) Find the measure of H.

4. (Ex.11.3.[8,9],p.456) Find the similarity dimension of:

   (a) the Sierpinski carpet in $\mathbf{R}^2$.

   (b) the Menger sponge in $\mathbf{R}^3$,

   (c) the Menger sponge in $\mathbf{R}^N$, $N > 3$.

5. (Ex.11.3.[8,9],p.456) Find the box dimension of:

   (a) the Sierpinski carpet in $\mathbf{R}^2$.

   (b) the Menger sponge in $\mathbf{R}^3$,

   (c) the Menger sponge in $\mathbf{R}^N$, $N > 3$.

6. (Ex.12.1.8,p.488) Hénon's area-preserving quadratic map is the iteration

$$\begin{aligned} x_{n+1} &= x_n \cos\alpha - (y_n - x_n^2)\sin\alpha \\ y_{n+1} &= x_n \sin\alpha + (y_n - x_n^2)\cos\alpha \end{aligned}$$

   where α is a fixed parameter satisfying $0 \le \alpha \le \pi$.

   (a) Show that the map is area-preserving.

   (b) Find the inverse mapping.

   (c) Plot at least 10,000 iterations from various (x_0, y_0) in the square $-1 \le x, y \le 1$ for various α including $\cos\alpha = 0.24$.

7. (Ex.12.1.9,p.488) The so-called *standard map* is the iteration

$$\begin{aligned} x_{n+1} &= x_n + y_{n+1} \pmod{2\pi}, \\ y_{n+1} &= y_n + k\sin x_n \pmod{2\pi}, \end{aligned}$$

   where $k \ge 0$ is a nonlinearity parameter. Notice that y_{n+1} must be computed first. It maps the square $S = [0, 2\pi] \times [0, 2\pi]$ into itself.

   (a) Show that the map is area-preserving for all k.

   (b) Plot at least 10,000 iterations from various (x_0, y_0) in the square S for each $k \in \{0, 0.5, 1, 2\}$.

8. (Ex.12.2.4,p.489) Find all fixed points for the Hénon map and show that they exist only if $a > a_0$ for some a_0, and then determine a_0.

9. (Ex.12.2.9,p.490) Let T be the Hénon map with parameters $a = 1.4$ and $b = 0.3$. Let $ABCD$ denote the quadrilateral with vertices $A = (-1.33, 0.42)$, $B = (1.32, 0.133)$, $C = (1.245, -0.14)$, and $D = (-1.06, -0.5)$.

   (a) Plot $ABCD$ and its image $T(ABCD)$. [HINT: represent the edges of $ABCD$ parametrically and note that T maps line segments to parabolic arcs.]

   (b) Prove that $T(ABCD) \subset ABCD$. Conclude that $ABCD$ is a trapping region for T.

10. (Ex.12.3.2,p.491)

    (a) Find the fixed points of the Rössler system, stating conditions on a, b, c under which they exist.

    (b) Fix $a = b = 0.2$ and plot a partial bifurcation diagram in terms of parameter c.

    (c) Find a trapping region for the system at your choice of a, b, c.