Mathematics 310: Foundations for Higher Mathematics
Model Solutions to the First Midterm Examination

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No collaboration, notes, texts, problem sets, or worked solutions are permitted. Please write your complete proofs in the blue notebooks. You have until the end of class to answer all of the following problems.

Problem 1: Let $a < b < c < d < e$ be real numbers. Express $[a,e] - ([b,c] \cup \{d\})$ as a union of intervals.

Solution 1: $[a,e] - ([b,c] \cup \{d\}) = [a,b) \cup (c,d) \cup (d,e]$. ■

Problem 2: Write the contrapositive of the following theorem:

$$[(\forall x \in \mathbf{R})(\forall \epsilon > 0)|f(x)| < \epsilon] \implies [(\forall x \in \mathbf{R}) f(x) = 0].$$

Solution 2: $[(\exists x \in \mathbf{R}) f(x) \neq 0] \implies [(\exists x \in \mathbf{R})(\exists \epsilon > 0)|f(x)| \geq \epsilon]$. ■

Problem 3: Prove that for $n \in \mathbf{N}$,

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Solution 3: Let $P(n)$ be the equation to be proved. We proceed by induction.

$P(1)$ is evidently true: $1^3 = \left(\frac{1(1+1)}{2}\right)^2$, after $n \leftarrow 1$.

Supposing $P(n)$ is true, we evaluate

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \frac{n^4 + 2n^3 + n^2 + 4n^3 + 12n^2 + 12n + 4}{4} \\
&= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4} \\
&= \frac{(n^2 + 3n + 2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2.
\end{aligned}$$

Thus $P(n) \implies P(n+1)$, and we conclude $(\forall n \in \mathbf{N}) P(n)$ by the principle of induction. ■

Problem 4: Suppose that $f : A \to A$ is a function. (a) Prove that if $f \circ f \circ f$ is injective then f is injective. (b) Prove that if $f \circ f \circ f$ is surjective then f is surjective.

Solution 4: (a) $f(x) = f(y) \implies f(f(x)) = f(f(y)) \implies f(f(f(x))) = f(f(f(y)))$, since f is a function and thus is single valued. But then $x = y$, since $f \circ f \circ f$ is injective. Thus $f(x) = f(y) \implies x = y$, so f is injective.

(b) Since $f \circ f \circ f$ is surjective, for every $b \in A$ there is some $a \in A$ such that $f(f(f(a))) = b$. But then $a' = f(f(a)) \in A$ satisfies $f(a') = b$, so for every $b \in A$ there is some $a' \in A$ such that $f(a') = b$. Hence f is surjective. ■

Problem 5: (a) Find f^{-1} and $f \circ f$ for the permutation f defined in (unsorted) two-line form by

$$\begin{pmatrix} 3 & 9 & 2 & 1 & 6 & 7 & 8 & 4 & 5 \\ 8 & 9 & 4 & 3 & 7 & 6 & 2 & 1 & 5 \end{pmatrix}$$

Write your answers in word form.

(b) Find the parity of the permutation $f \circ f$ of part a.

Solution 5: First sort the permutation into the more familiar sorted two-line form:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 8 & 1 & 5 & 7 & 6 & 2 & 9 \end{pmatrix}$$

Next, exchange the two lines and sort the top line to find the inverse:

$$\begin{pmatrix} 3 & 4 & 8 & 1 & 5 & 7 & 6 & 2 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 1 & 2 & 5 & 7 & 6 & 3 & 9 \end{pmatrix}$$

Thus $f^{-1} = 481257639$ in word form. To compute $f \circ f$, mimic functional digraph form by duplicating the sorted two-line form below itself, then sorting the lower two so that the third line is a copy of the second:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 8 & 1 & 5 & 7 & 6 & 2 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 8 & 1 & 5 & 7 & 6 & 2 & 9 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 8 & 1 & 5 & 7 & 6 & 2 & 9 \\ 3 & 4 & 8 & 1 & 5 & 7 & 6 & 2 & 9 \\ 8 & 1 & 2 & 3 & 5 & 6 & 7 & 4 & 9 \end{pmatrix}$$

We now have the word form $f \circ f = 812356749$ on the bottom line.

(b) The parity of a permutation is the parity of the number of transpositions needed to sort it to the identity. Whatever that number is for f, twice that number will sort $f \circ f$ to the identity, so the parity of $f \circ f$ is even. ■

Problem 6: Prove Pascal's formula: for all $n \in \mathbf{N}$ and all $k \in \mathbf{N}$ with $k < n$,

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Solution 6: Expand using the definitions, and find common denominators:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} = \frac{(n-1)!k}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

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