**QUIZ 6**

*Show your work not just your final answer*

(1) Let $M_2$ be the vector space of $2 \times 2$ matrices with the inner product defined as usual: For $A, B \in M_2$, $\langle A, B \rangle = \text{tr}(A^T B)$ where tr stands for the trace. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and let $W$ be the subspace spanned by $I, A$. Calculate the orthogonal projection of $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to $W$.

(2) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2. Define an inner product by,

$$\langle p(t), q(t) \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Starting with the standard basis $1, t, t^2$, use the Gram-Schmidt process to find an orthogonal basis.

(3) Show that for a positive integer $n$,

$$\int_0^1 x^n e^x dx \leq \sqrt{\frac{e^2 - 1}{4n + 2}}.$$

(Hint: Use the Cauchy-Schwartz inequality.)

(4) Find an orthogonal matrix $P$ and a diagonal matrix $D$ such that $PDP^T = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$.

(5) Let $B$ be an $n \times n$ symmetric matrix with $B^2 = B$. For any vector $\mathbf{x} \in \mathbb{R}^n$, show that $(\mathbf{x} - B\mathbf{x}) \cdot B\mathbf{x} = 0$.

(6) Find a $3 \times 3$ symmetric matrix $A$ so that

$$\mathbf{x}^T A\mathbf{x} = 4x_3^2 - 2x_1x_2 + 4x_2x_3$$

for any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(7) Find a change of coordinates $\mathbf{x} = P\mathbf{y}$ so that the quadratic form $2x_1^2 + 6x_1x_2 - 6x_2^2$ has no cross product terms.

(8) Let $A$ be an $n \times n$ invertible matrix. Show that $A^TA$ is positive definite.

(9) Find the singular values of the matrix $\begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}$.

(10) Find a singular value decomposition for the matrix,

$$A = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}$$