QUIZ 5

Show your work not just your final answer

(1) Solve the initial value problem,

$$x_1'(t) = x_1(t) - 2x_2(t)$$
$$x_2'(t) = 3x_1(t) - 4x_2(t)$$

for $t \geq 0$ with $x_1(0) = 3, x_2(0) = 2$.

(2) Let W be a subspace of $\mathbb{R}^n$.

(a) Show that $W^\perp$, the set of elements of $\mathbb{R}^n$ orthogonal to every element of W, is a susbspace of $\mathbb{R}^n$.

(b) Show that $(W^\perp)^\perp = W$ by showing that every element of W is in $(W^\perp)^\perp$ and conversely, every element of $(W^\perp)^\perp$ is in W.

(3) Show that

$$\mathbf{u} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

form an orthogonal basis for $\mathbb{R}^3$ and write $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

(4) Let L be a line in $\mathbb{R}^2$ through the origin. We define a map $R_L : \mathbb{R}^2 \to \mathbb{R}^2$ called *reflection* by $R_L(\mathbf{x}) = 2 \cdot \text{proj}_L(\mathbf{x}) - \mathbf{x}$ for any vector $\mathbf{x} \in \mathbb{R}^2$. Show that R_L is a linear transformation. Further show that R_L^2 (composition of R_L with itself) is the identity.

(5) Let A be an $m \times n$ matrix. Prove that every vector $\mathbf{x} \in \mathbb{R}^n$ can be written in the form $\mathbf{x} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p} \in \text{Row } A, \mathbf{u} \in \text{Nul } A$.

(6) Find an orthogonal basis for the column space of the matrix,

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$

(7) Let A be an $m \times n$ matrix. Show that $\text{Nul}\, A = \text{Nul}\, A^T A$.

(8) Let U be a 2×2 orthogonal matrix with determinant one. Show that,

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some θ.

(9) Find a least square solution of $A\mathbf{x} = \mathbf{b}$ where,

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

(10) Decide whether the following are true or false (with justifications). The underlying vector space is $\mathbb{R}^n$ with the usual inner product.

(a) It two vectors are orthogonal, they are linearly independent.

(b) If W is a subspace, W and $W^\perp$ have only the zero vector in common.

(c) If W is a subspace, then $\|\text{proj}_W \mathbf{v}\|^2 + \|\mathbf{v} - \text{proj}_W \mathbf{v}\|^2 = \|\mathbf{v}\|^2$.