ANSWERS TO QUIZ 6

Show your work not just your final answer

(1) Let M_2 be the vector space of 2×2 matrices with the inner product defined as usual, which I recall. For $A, B \in M_2$, $\langle A, B\rangle = \text{tr}(A^T B)$ where tr stands for the trace. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and W be the subspace spanned by I, A. Calculate the orthogonal projection of $E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ to W.

Solution. There are two ways to do this. One is to find an orthogonal basis for W using Gram-Schmidt process and then appealing to the formula for projection. (The formula does not apply without changing into an orthogonal basis, a mistake many of you have made in home work.) The other method is to note that $\text{proj}_W E = aI + bA$ for some scalars a, b and $E - aI - bA$ is orthogonal to both I and A to solve for a, b. We will follow the latter here.

So, we calculate

$$\langle I, E - aI - bA\rangle = \text{tr}(E - aI - bA) = 2 - 2a - 2b = 0.$$

$$\langle A, E - aI - bA\rangle = \text{tr}(A^T E - aA^T - bA^T A) = 2 - 2a - 4b = 0.$$

This gives $a = 1, b = 0$ and so $\boxed{\text{proj}_W(E) = I}$. □

(2) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2. Define an inner product by, $\langle p(t), q(t)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$. Starting with the standard basis $1, t, t^2$, use Gram-Schmidt process to find an orthogonal basis.

Solution. Gram-Scmidt says, keep 1 as it is and change first t to $t - \frac{\langle 1,t\rangle}{\langle 1,1\rangle}1$. $\langle 1, t\rangle = 3$ and $\langle 1, 1\rangle = 3$, thus our second basis is $u = t - 1$. Next, we change t^2 to, $t^2 - \frac{\langle 1,t^2\rangle}{\langle 1,1\rangle}1 - \frac{\langle u,t^2\rangle}{\langle u,u\rangle}(t-1)$. One easily calculates, $\langle 1, t^2\rangle = 5, \langle u, t^2\rangle = 4, \langle u, u\rangle = 2$. Thus, we may take the third to be $v = t^2 - \frac{5}{3} - 2(t-1) = t^2 - 2t + \frac{1}{3}$.

Thus an orthogonal basis is,

$$\boxed{1, t-1, t^2-2t-\frac{1}{3}}.$$

□

(3) Show that for a positive integer n,

$$\int_0^1 x^n e^x dx \leq \sqrt{\frac{e^2-1}{4n+2}}.$$

(Hint: Use Cauchy-Scwartz inequality)

Solution. Use Cauchy-Scwarz inequality. We consider the vector space $C^0[0,1]$, set of continuous function on the unit closed interval with the inner product $\langle f, g\rangle = \int_0^1 fgdx$. Then, by CS inequality,

$$\begin{aligned}
\int_0^1 x^n e^x dx &= \langle x^n, e^x\rangle \\
&\leq \sqrt{\langle x^n, x^n\rangle\langle e^x, e^x\rangle} \\
&= \sqrt{\int_0^1 x^{2n}dx \int_0^1 e^{2x}dx} \\
&= \sqrt{\frac{1}{2n+1}\frac{e^2-1}{2}} \\
&= \sqrt{\frac{e^2-1}{4n+2}}
\end{aligned}$$

□

(4) Find an orthogonal matrix P and a diagonal matrix D such that $PDP^T = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$.

Solution. We first calculate the eigenvalues. The characteristic polynomial is $\lambda^2 - 2\lambda - 24 = (\lambda-6)(\lambda+4)$. So, the eigen values are $6, -4$. Easy to check that unit eigenvector corresponding to 6 is $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and the one corresponding to -4

is, $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Thus,

$$\boxed{P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}}$$

□

(5) Let B be an $n \times n$ symmetric matrix with $B^2 = B$. For any vector $\mathbf{x} \in \mathbb{R}^n$, show that $((\mathbf{x} - B\mathbf{x}) \cdot B\mathbf{x}) = 0$.

Solution. We calculate:

$$\begin{aligned} ((\mathbf{x} - -B\mathbf{x}) \cdot B\mathbf{x}) &= (\mathbf{x} \cdot B\mathbf{x}) - (B\mathbf{x} \cdot B\mathbf{x}) \\ &= \mathbf{x}^T B\mathbf{x} - (B\mathbf{x})^T B\mathbf{x} \\ &= \mathbf{x}^T B\mathbf{x} - \mathbf{x}^T B^T B\mathbf{x} \\ &= \mathbf{x}^T B\mathbf{x} - \mathbf{x}^T B\mathbf{x} = 0 \end{aligned}$$

The last step, since $B^T B = BB = B^2 = B$. □

(6) Find a 3×3 symmetric matrix A so that for any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{x}^T A\mathbf{x} = 4x_3^2 - 2x_1x_2 + 4x_2x_3$.

Solution. See example 2 on page 403.

$$\boxed{A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}}$$

□

(7) Find a change of coordinates $\mathbf{x} = P\mathbf{y}$ so that the quadratic form $2x_1^2 + 6x_1x_2 - 6x_2^2$ has no cross product terms.

Solution. As before, the associated matrix is $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. Its characteristic polynomial is $\lambda^2 + 4\lambda - 21$ and so the eigenvalues are $3, -7$. Then, one can calculate corresponding unit

eigenvectors as

$$\mathbf{u} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}.$$

Taking $P = [\mathbf{u}\ \mathbf{v}]$, and $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$, we have $P^TAP = D$. So, if $\mathbf{x} = P\mathbf{y}$, we get,

$$\mathbf{y}^T D\mathbf{y} = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{x}^T A\mathbf{x}.$$

The last term corresponds to our quadratic form and the first term corresponds to the quadratic form $3y_1^2 - 7y_2^2$. □

(8) Let A be an $n \times n$ invertible matrix. Show that A^TA is positive definite.

Solution. See Theorem 5 on page 407. Also see page 418.

According to the above theorem, we only need to show that the eigenvalues of $B = A^TA$ are all positive. So, let λ be an eigenvalue of B and $\mathbf{u}$ be a corresponding eigenvector. Then we have $B\mathbf{u} = \lambda\mathbf{u}$. This gives, writing $\mathbf{v} = A\mathbf{u}$ and noting $\mathbf{v} \neq \mathbf{0}$ since A is invertible,

$$\begin{aligned} 0 &< \mathbf{v}^T\mathbf{v} \\ &= \mathbf{u}^T A^T A\mathbf{u} \\ &= \lambda\mathbf{u}^T\mathbf{u} \end{aligned}$$

This says λ multiplied by a positive number is positive and thus λ must be positive. □

(9) Find the singular values of the matrix $A = \begin{bmatrix} 3 & 0 \\ 8 & 3 \end{bmatrix}$.

Solution. $A^TA = \begin{bmatrix} 73 & 24 \\ 24 & 9 \end{bmatrix}$ and its characteristic polynomial is $(\lambda - 1)(\lambda - 81)$. So, the singular values of A are $9, 1$. □

(10) Find a singular value decomposition for the matrix,

$$A = \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix}$$

Solution. We first calculate

$$A^TA = \begin{bmatrix} 7 & 5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix}$$

One next calculates its characteristic polynomial to be $\lambda^2 - 100\lambda + 900$ and thus its eigenvalues are $90 \geq 10$. So, the singular values are $\sigma_1 = 3\sqrt{10} \geq \sigma_2 = \sqrt{10}$. Thus the rank of $A = 2$.

Next we calculate the corresponding unit eigenvectors to be $\mathbf{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$. Thus, we have,

$$\boxed{V = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}, \Sigma = \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}}$$

Finally, we take $\mathbf{u}_1 = \frac{A\mathbf{v}_1}{3\sqrt{10}}, \mathbf{u}_2 = \frac{A\mathbf{v}_1}{\sqrt{10}}$. So,

$$\mathbf{u}_1 = \begin{bmatrix} 15/\sqrt{5} \\ 15/\sqrt{5} \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 5/\sqrt{5} \\ -5\sqrt{5} \\ 0 \end{bmatrix}.$$

We extend these to an orthonormal basis for $\mathbb{R}^3$ and one can see that $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is one such. Thus, we may take,

$$\boxed{U = \begin{bmatrix} 15/\sqrt{5} & 5/\sqrt{5} & 0 \\ 15/\sqrt{5} & -5/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Then we have $A = U\Sigma V^T$. □