ANSWERS TO QUIZ 5

Show your work not just your final answer

(1) Solve the initial value problem,

$$x_1'(t) = x_1(t) - 2x_2(t)$$
$$x_2'(t) = 3x_1(t) - 4x_2(t)$$

for $t \geq 0$ with $x_1(0) = 3, x_2(0) = 2$.

Solution: Writing $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$, our equation is, $\mathbf{x}'(t) = A\mathbf{x}(t)$. The characteristic polynomial of A is $\lambda^2 + 3\lambda + 2$ and thus the eigenvalues are $-1, -2$. Corresponding eigenvctors are $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Writing $P = [\mathbf{u}\ \mathbf{v}]$, we have $P^{-1}AP = D$ where $D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$. Let $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$. Then,

$$\mathbf{y}'(t) = P^{-1}\mathbf{x}'(t) = P^{-1}A\mathbf{x}(t) = DP^{-1}\mathbf{x}(t) = D\mathbf{y}(t)$$

So, we have *uncoupled* the equations, to get,

$$y_1'(t) = -y_1(t)$$
$$y_2'(t) = -2y_2(t)$$

The solutions are thus, $y_1(t) = Ae^{-t}, y_2(t) = Be^{-2t}$ for constants A, B. $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. So, $\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. So, we get $A = 5, B = -1$. Thus

$$\mathbf{x}(t) = P\mathbf{y}(t) = P\begin{bmatrix} 5e^{-t} \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} 5e^{-t} - 2e^{-2t} \\ 5e^{-t} - 3e^{-2t} \end{bmatrix}.$$

(2) Let W be a subspace of $\mathbb{R}^n$.

(a) Show that $W^\perp$, the set of elements of $\mathbb{R}^n$ orthogonal to every element of W, is a susbspace of $\mathbb{R}^n$.

Solution: If $\mathbf{u}, \mathbf{v} \in W^\perp$, then for any $\mathbf{w} \in W$, we have,

$$\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} = 0.$$

Similarly, for any scalar $c \in \mathbb{R}$,

$$\mathbf{w} \cdot (c\mathbf{u}) = c(\mathbf{w} \cdot \mathbf{u}) = 0.$$

This shows $W^\perp$ is a subspace of $\mathbb{R}^n$.

(b) Show that $(W^\perp)^\perp = W$ by showing that every element of W is in $(W^\perp)^\perp$ and conversely, every element of $(W^\perp)^\perp$ is in W.

Solution: If $\mathbf{w} \in W$, for any element $\mathbf{z} \in W^\perp$, we have $\mathbf{w} \cdot \mathbf{z} = 0$ by definition and thus $\mathbf{w} \in (W^\perp)^\perp$.
Conversely, let $\mathbf{z} \in (W^\perp)^\perp$. Then we can write $\mathbf{z} = \mathbf{u} + \mathbf{v}$ where $\mathbf{u} = \text{proj}_W \mathbf{z}$ and $\mathbf{v}$ is orthogonal to W. (See Theorem 8, page 350). So, $\mathbf{v} \in W^\perp$ and thus $\mathbf{z} \cdot \mathbf{v} = 0$. Thus,

$$\mathbf{v} \cdot \mathbf{v} = \mathbf{v} \cdot (\mathbf{z} - \mathbf{u}) = \mathbf{v} \cdot \mathbf{z} - \mathbf{v} \cdot \mathbf{u} = 0.$$

So, $\mathbf{v} = 0$ and so $\mathbf{z} = \mathbf{u} \in W$.

(3) Show that

$$\mathbf{u} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$

form an orthogonal basis for $\mathbb{R}^3$ and write $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Solution: These form an orthogonal set is immediate and then by Theorem 4, page 340, they are linearly independent. So, they form a basis of $\mathbb{R}^3$ ($\dim \mathbb{R}^3 = 3$). If we write $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}$, then we know that $c_1 = \frac{(\mathbf{x} \cdot \mathbf{u})}{(\mathbf{u} \cdot \mathbf{u})}$ and similarly for c_2, c_3. So, $c_1 = \frac{24}{18} = \frac{4}{3}$. Similar calculation will yield, $c_2 = \frac{1}{3}$ and $c_3 = \frac{1}{3}$.

(4) Let L be a line in $\mathbb{R}^2$ through the origin. We define a map $R_L : \mathbb{R}^2 \to \mathbb{R}^2$ called *reflection* by $R_L(\mathbf{x}) = 2 \cdot \text{proj}_L(\mathbf{x}) - \mathbf{x}$ for

any vector $\mathbf{x} \in \mathbb{R}^2$. Show that R_L is a linear transformation. Further show that R_L^2 (composition of R_L with itself) is the identity.

Solution: We check the two properties for linear transformation.

$$\begin{aligned} R_L(\mathbf{x}+\mathbf{y}) &= 2\,\mathrm{proj}_L(\mathbf{x}+\mathbf{y}) - (\mathbf{x}+\mathbf{y}) \\ &= (2\,\mathrm{proj}_L(\mathbf{x}) + 2\,\mathrm{proj}_L(\mathbf{y})) - (\mathbf{x}+\mathbf{y}) \\ &= (2\,\mathrm{proj}_L(\mathbf{x}) - \mathbf{x}) + (2\,\mathrm{proj}_L(\mathbf{y}) - \mathbf{y}) \\ &= R_L(\mathbf{x}) + R_L(\mathbf{y}). \end{aligned}$$

It is equally straightforward to check that for any $c \in \mathbb{R}$, $R_L(c\mathbf{x}) = cR_L(\mathbf{x})$.

Next we calculate R_L^2.

$$\begin{aligned} R_L^2(\mathbf{x}) &= R_L(2\,\mathrm{proj}_L(\mathbf{x}) - \mathbf{x}) \\ &= (4\,\mathrm{proj}_L(\mathrm{proj}_L(\mathbf{x})) - 2\,\mathrm{proj}_L(\mathbf{x})) - (2\,\mathrm{proj}_L(\mathbf{x}) - \mathbf{x}) \end{aligned}$$

Since $\mathrm{proj}_L(\mathbf{x}) \in L$, we see that $\mathrm{proj}_L(\mathrm{proj}_L(\mathbf{x})) = \mathrm{proj}_L(\mathbf{x})$. Thus,

$$R_L^2(\mathbf{x}) = 4\,\mathrm{proj}_L(\mathbf{x}) - 2\,\mathrm{proj}_L(\mathbf{x}) - 2\,\mathrm{proj}_L(\mathbf{x}) + \mathbf{x} = \mathbf{x}.$$

(5) Let A be an $m \times n$ matrix. Prove that every vector $\mathbf{x} \in \mathbb{R}^n$ can be written in the form $\mathbf{x} = \mathbf{p} + \mathbf{u}$, where $\mathbf{p} \in \mathrm{Row}\,A$, $\mathbf{u} \in \mathrm{Nul}\,A$.

Solution: Writing $R = \mathrm{Row}\,A$, one has $\mathbf{x} = \mathrm{proj}_R(\mathbf{x}) + \mathbf{u}$, where $\mathbf{u} \in R^\perp$. By theorem 3, page 337, $R^\perp = \mathrm{Nul}\,A$ and thus $\mathbf{u} \in \mathrm{Nul}\,A$. Denote by $\mathbf{p} = \mathrm{proj}_R(\mathbf{x})$ and then $\mathbf{p} \in R$.

(6) Find an orthogonal basis for the column space of the matrix,

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$

Solution: We appeal to Gram-Schmidt process. Denote by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ the three columns in the given order. Since $\mathbf{u} \neq 0$, we keep it as is and replace $\mathbf{v}$ by $\mathbf{v}' = \mathbf{v} - \frac{(\mathbf{v}\cdot\mathbf{u})}{(\mathbf{u}\cdot\mathbf{u})}\mathbf{u}$. $(\mathbf{v}\cdot\mathbf{u}) =$

16, $(\mathbf{u}\cdot\mathbf{u}) = 4$, so this becomes, $\mathbf{v}' = \mathbf{v} - 4\mathbf{u}$. Next, we replace $\mathbf{w}$ by $\mathbf{w}' = \mathbf{w} - \frac{(\mathbf{w}\cdot\mathbf{u})}{(\mathbf{u}\cdot\mathbf{u})}\mathbf{u} - \frac{(\mathbf{w}\cdot\mathbf{v}')}{(\mathbf{v}'\cdot\mathbf{v}')}\mathbf{v}'$. One easily calculates,

$$\mathbf{v}' = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{w}' = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

and $\mathbf{u}, \mathbf{v}', \mathbf{w}'$ form an orthogonal basis for Col A.

(7) Let A be an $m \times n$ matrix. Show that Nul A = Nul A^TA.

Solution: Let $\mathbf{x} \in$ Nul A. Then $A\mathbf{x} = 0$ and then $A^TA\mathbf{x} = 0$, so $\mathbf{x} \in$ Nul A^TA. Conversely, let $\mathbf{x} \in$ Nul A^TA. Then $A^TA\mathbf{x} = 0$ and then,

$$\begin{aligned} 0 &= \mathbf{x}^TA^TA\mathbf{x} \\ &= (A\mathbf{x})^T(A\mathbf{x}) \\ &= (A\mathbf{x}\cdot A\mathbf{x}) \end{aligned}$$

This implies $A\mathbf{x} = 0$ and thus $\mathbf{x} \in$ Nul A, proving what we need.

(8) Let U be a 2×2 orthogonal matrix with determinant one. Show that,

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some θ.

Solution: Write $U = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ for some $A, B, C, D \in \mathbb{R}$. Since the column vectors are of unit length, we have $A^2 + B^2 = 1, C^2 + D^2 = 1$ and then we can write $A = \cos\theta, B = \sin\theta$ and $C = -\sin\phi, D = \cos\phi$ for some θ, ϕ. The column vectors are orthogonal says,

$$\begin{aligned} 0 &= -\cos\theta\sin\phi + \sin\theta\cos\phi \\ &= \sin(\theta - \phi) \end{aligned}$$

So, $\theta = \phi$ or $\theta = \phi + \pi$. In the latter case, easy to check that the determinant is -1 since $\cos(\phi + \pi) = -\cos\phi$ and $\sin(\phi + \pi) = -\sin\phi$. So, $\theta = \phi$ and thus U is of the required form.

(9) Find a least square solution of $A\mathbf{x} = \mathbf{b}$ where,

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

Solution: We use Theorem 13, page 363. We compute A^TA and $A^T\mathbf{b}$. You can check that,

$$A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}, \ A^T\mathbf{b} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Then, we wish to solve $A^TA\mathbf{x} = A^T\mathbf{b}$ and one gets $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the unique solution.

(10) Decide whether the following are true or false (with justifications). The underlying vector space is $\mathbb{R}^n$ with the usual inner product.

(a) It two vectors are orthogonal, they are linearly independent.

Solution: False. The zero vector is orthogonal to any vector, but will not be linearly independent.

(b) If W is a subspace, W and $W^\perp$ have only the zero vector in common.

Solution: True. If $\mathbf{u}$ belongs to both W and $W^\perp$, then $(\mathbf{u} \cdot \mathbf{u}) = 0$, which implies $\mathbf{u} = 0$.

(c) If W is a subspace, then $\| \operatorname{proj}_W \mathbf{v}\|^2 + \|\mathbf{v} - \operatorname{proj}_W \mathbf{v}\|^2 = \|\mathbf{v}\|^2$.

Solution: True. Write $\operatorname{proj}_W \mathbf{v} = \mathbf{u}$. Then, $\mathbf{u}$ and $\mathbf{v} - \mathbf{u}$ are orthogonal. Therefore,

$$\begin{aligned} \|\mathbf{v}\|^2 &= (\mathbf{v} \cdot \mathbf{v}) \\ &= (\mathbf{u} + (\mathbf{v} - \mathbf{u})) \cdot (\mathbf{u} + (\mathbf{v} - \mathbf{u})) \\ &= (\mathbf{u} \cdot \mathbf{u}) + 2(\mathbf{u} \cdot (\mathbf{v} - \mathbf{u})) + ((\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u})) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v} - \mathbf{u}\|^2. \end{aligned}$$