QUIZ 4

Show your work not just your final answer

(1) Let f, g, h be functions from $\mathbb{R}$ to itself which are linearly independent in the vector space of all such functions from reals to itself. Define elements of the vector space of signals by $F(k) = f(k), G(k) = g(k), H(k) = h(k), k \in \mathbb{Z}$. Are F, G, H linearly independent?

No, in general they are not. I will give an example to show this (there is nothing unique about this example and you may have a different one, which is fine.)

Take $f(x) = \cos \pi x, g(x) = \sin \pi x, h = \sin 2\pi x$. I claim they are linearly independent in the vector space of all functions from the reals to itself. If $c_1 f + c_2 g + c_3 h = 0$, putting $x = 0$, one gets $c_1 = 0$. So, our relation becomes $c_2 g + c_3 h = 0$. Putting $x = 1/2$, one gets $c_2 = 0$, which immediately shows $c_3 = 0$. But notice that $G(k) = \sin k\pi = 0, H(k) = \sin 2k\pi = 0$ for any $k \in \mathbb{Z}$ and thus F, G, H are not linearly independent.

(2) Show that if P is a stochastic matrix, so is P^2.

Write $P = [a_{ij}]$ as usual. If $P^2 = [b_{ij}]$, we have $b_{ij} = \sum_k a_{ik} a_{kj}$. Since $a_{ij} \geq 0$, we see that $b_{ij} \geq 0$. We need to check that the column sums up to1 for P^2 next, but this is same as checking $\mathbf{v}P^2 = \mathbf{v}$ where $\mathbf{v} = [1\,1\,1 \cdots 1]$. But, $\mathbf{v}P = \mathbf{v}$ since P is a stochastic matrix and thus, $\mathbf{v}P^2 = \mathbf{v}PP = \mathbf{v}P = \mathbf{v}$.

(3) Decide which of the following are true (with justifications always).

(a) If A is a square matrix and λ, μ are two eigenvalues of A then $\lambda\mu$ is an eigenvalue of A^2.
False. Take A, a 2×2 diagonal matrix with $2, 3$ on the main diagonal. Then $2, 3$ are its eigenvalues and A^2 is the diagonal matrix with $4, 9$ on the main diagonal and thus its eigenvalues are $4, 9$, neither equal to $6 = 2 \times 3$.

(b) An eigenvalue of a matrix A is also an eigenvalue of A^T.
True. λ is an eigenvalue of A is same as saying $\det(A - \lambda I) = 0$. Since $\det B = \det B^T$, this means $\det(A^T -$

$\lambda I) = 0$ (transpose of the identity matrix is still the identity matrix). Thus λ is an eigenvalue of A^T.

(c) An $n \times n$ matrix with n linearly independent eigenvectors is invertible.
False. The zero matrix has n linearly independent eigenvectors (for example, the standard basis $\mathbf{e}_i$), but clearly not invertible.

(4) Let $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ be a polynomial and A a square matrix. Define $p(A)$ to be $c_0 I + c_1 A + \cdots + c_n A^n$. Show that if λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue of $p(A)$.

Let $\mathbf{v}$ be an eigenvector corresponding to the eigenvalue λ. Then, for any m, one has $A^m \mathbf{v} = \lambda^m \mathbf{v}$. Thus,

$$\begin{aligned} p(A)\mathbf{v} &= (c_0 I + c_1 A + \cdots + c_n A^n)\mathbf{v} \\ &= c_0 I\mathbf{v} + c_1 A\mathbf{v} + \cdots + c_n A^n \mathbf{v} \\ &= c_0 \mathbf{v} + c_1 \lambda \mathbf{v} + \cdots + c_n \lambda^n \mathbf{v} \\ &= (c_0 + c_1 \lambda + \cdots + c_n \lambda^n)\mathbf{v} \\ &= p(\lambda)\mathbf{v} \end{aligned}$$

So, $p(\lambda)$ is an eigenvalue of $p(A)$ with the same eigenvector $\mathbf{v}$.

(5) Let

$$A = \begin{bmatrix} 0.4 & -0.3 \\ 0.4 & 1.2 \end{bmatrix}$$

Calculate $\lim_{k\to\infty} A^k$.

We first calculate the eigenvalues and they are $1, 0.6$ (we have done this a lot of times, but you should show your work). Then we calculate corresponding eigenvectors and again, you should know how. These are (eigenvectors are not unique),

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

So, as usual, we let $P = [\mathbf{v}_1\ \mathbf{v}_2]$ and then, $P^{-1}AP = D$, where D is the diagonal matrix with $1, 0.6$ in the main diagonal. Then, for any integer k, we have $P^{-1}A^kP = D^k$ and D^k is the diagonal matrix with $1^k = 1, 0.6^k$ in the main diagonal. Taking limits, we get, $P^{-1} \lim_k A^k P = \lim_k D^k$. Since $0.6^k \to 0$ as $k \to \infty$, $\lim_k D^k$ is the diagonal matrix with $1, 0$ in the main

diagonal. So,

$$\lim_k A^k = P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

One checks $P^{-1} = \begin{bmatrix} -0.5 & -0.75 \\ 0.5 & 0.25 \end{bmatrix}$ and then

$$P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} -0.5 & -0.75 \\ 1 & 1.5 \end{bmatrix}$$

(6) If A is an $n \times n$ real matrix such that $A = A^T$ (such matrices are called *symmetric*), show that its eigenvalues are real.

Let λ be an eigenvalue and $\mathbf{v}$ corresponding eigenvector, possibly complex. We have $A\mathbf{v} = \lambda\mathbf{v}$. Taking complex conjugates (A is real), we get $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. So, we get

$$\begin{aligned} \overline{\mathbf{v}}^T \lambda \mathbf{v} &= \overline{\mathbf{v}}^T A \mathbf{v} \\ &= \overline{\mathbf{v}}^T A^T \mathbf{v} \\ &= (A\overline{\mathbf{v}})^T \mathbf{v} \\ &= \overline{\lambda}\overline{\mathbf{v}}^T \mathbf{v} \end{aligned}$$

So, $(\lambda - \overline{\lambda})\overline{\mathbf{v}}^T\mathbf{v}$. Since $\mathbf{v} \neq= 0$, $\overline{\mathbf{v}}^T\mathbf{v} > 0$ and thus $\lambda = \overline{\lambda}$, which says λ (and thus $\mathbf{v}$) is real.

(7) Find an invertible matrix P with main diagonal elements $2, 1$ and a matrix C of the form,

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

such that

$$PCP^{-1} = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} = A$$

See Theorem 9, page 301. We calculate the eigenvalues of A. The characteristic polynomial is $\lambda^2 - 8\lambda + 17$ and thus the two eigenvalues are $4 \pm i$. We calculate an eigenvector corresponding to $4 - i$. An eigenvector (always remember eigenvectors are not quite unique) can easily be checked to be $\mathbf{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$. Thus, we may take $P = [\operatorname{Re}\mathbf{v}\ \operatorname{Im}\mathbf{v}]$ and then,

$$P = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$$

(8) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given as $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

Find an ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ with all coordinates being integers, so that the matrix of T relative to $\mathcal{B}$ (for both $\mathbb{R}^3$), is diagonal.

Since A is upper triangular, its eigenvalues are $1, 3$ and 4 and they being distinct, their eigenvectors form a basis for $\mathbb{R}^3$. One calculates (I do not show the calculations, but you should), the eigenvectors in the order of the above eigenvalues to be,

$$\mathbf{v}_1 = \mathbf{e}_1, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 17 \\ 6 \\ 3 \end{bmatrix}$$

So, we can take $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. See theorem 8 on page 293.

(9) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that the characteristic polynomial is $p(t) = \lambda^2 - (a + d)\lambda + \det A$. Then show that $p(A) = A^2 - (a+d)A + (\det A)\, I = 0$.

We have $\det A = ad - bc$ and $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$.

So we get,

$$\begin{aligned} p(A) &= A^2 - (a+d)A + (\det A)\, I \\ &= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} - \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \end{aligned}$$

The last can easily seen to be the zero matrix by elementary algebra.

(10) Find all real eigenvalues and the corresponding eigenvectors for

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Either by direct calculation or noting that the column space is one dimensional, since the second and third column is a

multiple of the first, one sees that the null space is two dimensional. Thus the matrix has 0 as an eigenvalue with multiplicity two and the third eigenvalue is 6. An eigenvector corresponding to 6 is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Eigenvectors corresponding to zero can be any basis for the null space. For example,

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

(11) Find the steady vector for

$$P = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

A steady-state vector is an eigenvector which is a probability vector for the eigenvalue 1 and thus a basis for the null space for $P - I = \begin{bmatrix} -0.2 & 0.5 \\ 0.2 & -0.5 \end{bmatrix}$. One easily sees that $\begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix}$ is the answer.

(12) Decide which of the following are true (with justifications)

(a) If A, B are two $n \times n$ matrices and λ is an eigenvalue for both, then it is an eigenvalue of $A + B$.
False. As an example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then both A, B have zero as an eigenvalue, but $A + B = I$ has only 1 as an eigenvalue.

(b) If A, B are two $n \times n$ matrices and $\mathbf{x}$ is an eigenvector for both, then $\mathbf{x}$ is also an eigenvector for $A + B$.
True. If $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{x} = \mu\mathbf{x}$, then $(A + B)\mathbf{x} = (\lambda + \mu)\mathbf{x}$.

(c) If A is an $n \times n$ matrix, then $A - I_n$ is invertible if 1 is not an eigenvalue of A.
True. If $A - I_n$ is not invertible, its null space must be non-zero and then any non-zero vector in the null space is an eigenvector for A with eigenvalue 1.