QUIZ 3

Show your work not just your final answer

(1) Compute $\det(A^5)$ for

$$A = \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & -2 \\ 0 & -1 & 4 \end{pmatrix}.$$

Solution: First compute $\det A$ by expansion in cofactors, using the first column:

$$\det \begin{pmatrix} 1 & 5 & 7 \\ 0 & 1 & -2 \\ 0 & -1 & 4 \end{pmatrix} = (1)(-1)^{1+1} \det \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix} = (1)(1)(2) = 2$$

Then use the multiplicative property of determinants, Theorem 6 on textbook page 175:

$$\boxed{\det(A^5) = (\det A)^5 = 2^5 = 32}$$

(2) Suppose that B is a 4×4 matrix with $\det B = 3$. Let $C = 2B^T$. Compute $\det C$.

Solution: From the textbook, Theorem 5 on p.174, $\det(B^T) = \det B$. By Theorem 3c, p.171, since the 4 columns of $2B^T$ are the rows of B each multiplied by 2,

$$\boxed{\det C = \det(2B^T) = 2^4 \det(B) = (16)(3) = 48.}$$

(3) Find the matrix of cofactors of the following matrix:

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 4 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution: Extract the ij minor A_{ij}, compute its determinant with the 2×2 determinant formula, then multiply by $(-1)^{i+j}$

to get the cofactor C_{ij}. Arrange them in a matrix as follows:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A_{11} & -\det A_{12} & \det A_{13} \\ -\det A_{21} & \det A_{22} & -\det A_{23} \\ \det A_{31} & -\det A_{32} & \det A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \det\begin{pmatrix} 4 & 0 \\ -1 & 1 \end{pmatrix} & -\det\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} & \det\begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix} \\ -\det\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \det\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} & -\det\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \\ \det\begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix} & -\det\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} & \det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 4 & -3 & -3 \\ 0 & 2 & 2 \\ 4 & -3 & 5 \end{bmatrix}}$$

NOTE: there is no "matrix of cofactors" command in Octave or MATLAB, but the command `det(A)*inv(A)'` calculates it as seen in Theorem 8, p.181 of the text.

(4) Find the area of the parallelogram with vertices $(0,0)$, $(3,3)$, $(8,4)$, and $(5,1)$.

Solution: Since one of the vertices is $(0,0)$, this paralellogram is defined by the two vectors $(3,3)$ and $(5,1)$. (We may check that the fourth vertex is $(3,3)+(5,1) = (8,4)$.) By Theorem 9 on p.182, its area is

$$\left|\det\begin{pmatrix} 3 & 5 \\ 3 & 1 \end{pmatrix}\right| = |(3)(1)-(5)(3)| = |-12| = \boxed{12.}$$

(5) Let V be the subset of the 4×4 matrices consisting of those with zeros at every main diagonal element. Is V a subspace?

Solution: Check the three Subspace properties:

i: V contains 0, since the zero matrix has all zeroes on the main diagonal.

ii: If $A \in V$ and $c \in \mathbf{R}$, then cA has all zeroes on the main diagonal, so $cA \in V$.

iii: If $A \in V$ and $B \in V$, then $A+B$ has all zeroes on the main diagonal, so $A+B \in V$.

Conclude that V is a subspace. $\boxed{\text{YES}}$.

(6) Which of these subsets of P_9, the polynomials $p(t)$ with degree 9 or less, is a subspace:

(a) $\{p(t) : p(0) = 17\}$

(b) $\{p(t) : \int_0^1 p(t)\,dt = 17\}$

(c) $\{p(t) : p'(0) = p(17)\}$

Solution:

(a) $\boxed{\text{NO}}$. Does not contain 0.

(b) $\boxed{\text{NO}}$. Does not contain 0.

(c) $\boxed{\text{YES}}$. Check the three Subspace properties:

i: contains $p(t) = 0$, since then $p'(t) = 0$ so $p'(0) = 0 = p(17)$.

ii: If $p'(0) = p(17)$ and $c \in \mathbf{R}$, then $cp'(0) = cp(17)$, so cp is in the subset.

iii: If $p'(0) = p(17)$ and $q'(0) = q(17)$ then $(p+q)'(0) = (p+q)(17)$, so $p+q$ is in the subset.

(7) Let

$$A = \begin{bmatrix} 2 & -2 & -1 \\ 8 & -9 & -6 \\ 10 & -7 & 5 \\ 2 & 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ k \end{bmatrix}$$

Find k such that $\mathbf{b}$ is in the column space of A.

Solution: By Theorem 3, textbook p.203, $\mathbf{b}$ is in the column space of A if and only if $A\mathbf{x} = \mathbf{b}$ has a solution. To find that solution, row reduct the augmented matrix $(A|\mathbf{b})$ to echelon

form:

$$
\begin{aligned}
(A|\mathbf{b}) &= \left[\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 8 & -9 & -6 & 2 \\ 10 & -7 & 5 & 3 \\ 2 & 2 & 3 & k \end{array}\right] \\
&\to \left[\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 3 & 10 & -2 \\ 0 & 4 & 4 & k-1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & -4 & k-9 \end{array}\right] \\
&\to \left[\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & k-17 \end{array}\right]
\end{aligned}
$$

This system will have a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ if and only if $k - 17 = 0$, namely if $\boxed{k = 17}$.

(8) Let $T : \mathbf{R}^5 \to \mathbf{R}^3$ be defined by

$$
T(x_1, x_2, x_3, x_4, x_5) = \begin{bmatrix} x_1 + 4x_4 - 2x_5 \\ 3x_1 + x_2 - 9x_5 \\ 5x_1 - x_2 + x_3 \end{bmatrix}.
$$

Find the dimension of the range of T and the dimension of the kernel of T.

Solution: First find the standard matrix A for T, such that $T(\mathbf{x}) = A\mathbf{x}$, then row reduce to echelon form to identify the

pivot and free variables:

$$A = \begin{bmatrix} 1 & 0 & 0 & 4 & -2 \\ 3 & 1 & 0 & 0 & 5 \\ 5 & -1 & 1 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & 4 & -2 \\ 0 & 1 & 0 & -12 & 11 \\ 0 & -1 & 1 & -20 & 10 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & 4 & -2 \\ 0 & 1 & 0 & -12 & 11 \\ 0 & 0 & 1 & -32 & 21 \end{bmatrix}$$

Evidently columns 1,2,3 are pivot columns and columns 4,5 are free-variable columns. Conclude that

$$\boxed{\dim \text{range T} = 3, \quad \dim \text{kernel T} = 2}$$

(9) Suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a set of linearly independent vectors. Which of the following sets of vectors is also linearly independent?
(a) $\{\mathbf{0}, \mathbf{v}, \mathbf{w}\}$
(b) $\{\mathbf{v}, \mathbf{w}\}$
(c) $\{\mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}\}$
(d) $\{\mathbf{w}\}$
(e) $\{\mathbf{u}, \mathbf{v} + \mathbf{w}\}$
(f) $\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$

Solution:
(a) $\boxed{\text{NO}}$. The zero vector makes the set linearly dependent.
(b) $\boxed{\text{YES}}$. A subset of a linearly independent set is linearly independent.
(c) $\boxed{\text{NO}}$. $\mathbf{u} + \mathbf{v}$ is obviously a linear combination of $\mathbf{u}, \mathbf{v}$. Apply Theorem 4, p.210 of the textbook.
(d) $\boxed{\text{YES}}$. A subset of a linearly independent set is linearly independent.
(e) $\boxed{\text{YES}}$. This requires proof, which is easiest done by the contrapositive. If $\{\mathbf{u}, \mathbf{v} + \mathbf{w}\}$ is not linearly independent, the there exist $a, b \in \mathbf{R}$ not both zero such that

$$0 = a\mathbf{u} + b(\mathbf{v} + \mathbf{w}) = a\mathbf{u} + b\mathbf{v} + b\mathbf{w}$$

But this is a linear dependence relation for $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, so that set must not be linearly independent.

(f) YES. This requires proof, which we will do directly. If $\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is not linearly independent, Suppose that $a, b, c \in \mathbf{R}$ satisfy

$$0 = a\mathbf{u} + b(\mathbf{u}+\mathbf{v}) + c(\mathbf{u}+\mathbf{v}+\mathbf{w}) = (a+b+c)\mathbf{u} + (b+c)\mathbf{v} + c\mathbf{w}.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a set of linearly independent vectors, we may conclude that

$$\begin{aligned} a+b+c &= 0 \\ b+c &= 0 \\ c &= 0 \end{aligned}$$

This is an upper triangular linear system which has the unique solution $a = b = c = 0$. Conclude that $\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is a linearly independent set.

(10) Let $S = \{1, 1+t, 1+t+t^2\}$ be a subset of $\mathbf{P}_2$, the polynomials of degree 2 or less.
(a) Is S a linearly independent set?
(b) Is S a basis for $\mathbf{P}_2$?

Solution:
(a) YES. Put $\mathbf{u} = 1$, $\mathbf{v} = t$, and $\mathbf{w} = t^2$. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a set of linearly independent vectors by Example 6 on p.211 of the textbook (it is a basis for $\mathbf{P}_2$). By part (f) of problem 9 above, $S = \{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w}\}$ is also linearly independent.
(b) YES. Since S is linearly independent, it remains to prove that S spans $\mathbf{P}_2$. But if $p \in \mathbf{P}_2$ is not in the span of S, then $\{1, 1+t, 1+t+t^2, p(t)\}$ is a set of 4 linearly independent vectors in the 3-dimensional space $\mathbf{P}_2$, which is impossible. Conclude that $\mathbf{P}_2 \subset \operatorname{span} S$, so $\mathbf{P}_2 = \operatorname{span} S$, so S is a basis for $\mathbf{P}_2$.

(11) Consider the ordered basis B of $\mathbf{R}^2$ consisting of the vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Find $\mathbf{x} \in \mathbf{R}^2$ whose coordinates with respect to this basis are

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 7 \end{bmatrix},$$

Solution: Expand $\mathbf{x}$ as a linear combination of the vectors in B:

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \implies \mathbf{x} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 17 \end{bmatrix}}$$

(12) Consider the following ordered basis B for the vector space of 2×2 lower triangular matrices:

$$B = \left\{ \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

Find the coordinates $[M]_B$ of

$$M = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$

with respect to this basis.

Solution: Write

$$[M]_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and solve for the unknowns x, y, z in the resulting equation:

$$\begin{aligned} M &= \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} = x\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + z\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} y+z & 0 \\ 2x-y & x+2z \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \implies \quad & y + z = 1 \\ & 2x - y = 2 \\ & x + 2z = -3 \end{aligned}$$

$$\implies \quad \begin{bmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\implies \quad \boxed{[M]_B = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix}}$$

The last step is done by row reduction to reduced echelon form of the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 2 & -1 & 0 & 2 \\ 1 & 0 & 2 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & -1 & -4 & 8 \\ 1 & 0 & 2 & -3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & -3 & 9 \\ 0 & -1 & -4 & 8 \\ 1 & 0 & 2 & -3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & -3 \\ 0 & 1 & 4 & -8 \\ 1 & 0 & 2 & -3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$