MIDTERM 2

Show your work not just your final answer

(1) Let $p_1(t) = 1 + t$, $p_2(t) = 1 - t$, $p_3(t) = 4$, $p_4(t) = t + t^2$ and $p_5(t) = 1 + 2t + t^2$ and let H be the subspace of $\mathbb{P}_5$ spanned by $S = \{p_1, p_2, p_3, p_4, p_5\}$. Use the methods of spanning theorem to find a subset of S which is a basis of H.

$p_1 \neq 0$ and thus linearly independent. p_1, p_2 are linearly independent. I will check this for completeness. If $c_1 p_1 + c_2 p_2 = 0$, we have $c_(1 + t) + c_2(1 - t) = 0$. Equating coefficients of powers of t, we get $c_1 + c_2 = 0, c_1 - c_2 = 0$. Together, we get $c_1 = c_2 = 0$.

$p_3 = 2(p_1 + p_2)$ and thus, we skip p_3. p_4 is not in the subspace generated by p_1, p_2, since this subspace has all polynomials of degree at most one and $\deg p_4 = 2$. Finally, $p_5 = p_1 + p_4$ and thus, we do not need p_5. So, p_1, p_2, p_4 form a basis of H.

(2) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be defined as $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$.

(a) Show that T is a linear transformation.

$$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = T(p) + T(q).$$

Similarly, for a $c \in \mathbb{R}$, we get,

$$T(cp) = \begin{bmatrix} (cp)(0)) \\ (cp)(1) \end{bmatrix} = \begin{bmatrix} cp(0) \\ cp(1) \end{bmatrix} = c \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = cT(p).$$

(b) Find an element $q(t) \in \mathbb{P}_2$, monic of degree 2 (that means $q(t) = t^2 + at + b$ for some $a, b \in \mathbb{R}$) such that $T(q) = 0$. Show that q forms a basis for the kernel of T.

Let $q = t^2 + at + b$ for some $a, b \in \mathbb{R}$. If $T(q) = 0$, we must have $q(0) = q(1) = 0$. The first gives $b = 0$ ans then the second gives $1 + a = 0$. So, $q = t^2 - t$. If $p(t)$ is in the kernel of T, writing $p = at^2 + bt + c$, we must have $p(0) = p(1) = 0$. The firs gives, $c = 0$ and the second

gives $a + b = 0$. So, $p = a(t^2 - t) = aq$. So, q forms a basis for the kernel of T.

(3) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be the basis of $\mathbb{R}^3$ consisting of

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Calculate $[\mathbf{x}]_{\mathcal{B}}$ where $\mathbf{x} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$.

Recall that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ means, $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$. We have done this many times. So, we write the augmented matrix, $[\mathbf{b}_1\, \mathbf{b}_2\, \mathbf{b}_3\, \mathbf{x}]$ and use row reductions to solve for the c_is.

This should give you $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$.

(4) Determine the dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ where,

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix is already in the echelon form, so the dimension of $\operatorname{Col} A$ is just the number of pivots, which is 4 (the pivots are in first, third, fourth and sixth columns). Since $\dim \operatorname{Col} A + \dim \operatorname{Nul} A = 6$, we get $\dim \operatorname{Nul} A = 2$.

(5) Let P be a 2×2 stochastic matrix with all entries positive. Show that 1 is an eigenvalue of P with multiplicity one.

Since $[1\,1]P = [1\,1]$ (this is just saying the columns of P add to 1), we get $[1\,1](P - I) = 0$ and thus $P - I$ is not invertible. Thus, $\det(P - I) = 0$, which says 1 is an eigenvalue. There are many ways of seeing that this has multiplicity one. Let me write one of them. Let a, d be the entries in the main diagonal. Then, $a + d$ is the sum of the two roots of the characteristic polynomial and so, if 1 is a double root, we get $a + d = 2$.

But, $a, d \geq 0$ and $a, d < 1$, since all entires of P are positive. So, $a + d < 2$.

(6) Find a basis for the solution space of the difference equation,

$$16y_{k+2} + 8y_{k+1} - 3y_k = 0.$$

We look at the auxiliary equation $16r^2 + 8r - 3 = 0$. Quadratic formula gives $r = \frac{1}{4}, -\frac{3}{4}$. Thus, a general solution is $y_k = c_1(\frac{1}{4})^k + c_2(-\frac{3}{4})^k$ for $c_i \in \mathbb{R}$. Thus the solution space is two dimensional with basis $\{y_k\} = \{(\frac{1}{4})^k\}$ and $\{y_k\} = \{(-\frac{3}{4})^k\}$.

(7) Find the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 4$ for the matrix,

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

This is just the dimension of the null space of $A - 4I$. So, we do our usual row reduction techniques to $A - 4I$.

$$\begin{aligned} A - 4I &= \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\to \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &\to \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The last has two pivots and thus $\dim \mathrm{Col}(A - 4I) = 2$ and so $\dim \mathrm{Nul}(A - 4I) = 2$.

(8) Find h so that the eigenspace corresponding to the eigenvalue $\lambda = 5$ for

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is two dimensional.

As before we calculate the dimension of the column space of $A - 5I$.

$$A - 5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & 0 & h-6 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

We see that if $h \neq 6$, the last matrix has three pivots (second, third and fourth columns) and so the dimension of column space is three and so the dimension of the null space is one. If $h = 6$, then the dimension of the column space is 2 and thus the required eigenspace has dimension 2.

(9) Calculate A^k where k is any positive integer and

$$A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$$

We first calculate the eigenvalues. The characteristic polynomial is $\lambda^2 - 3\lambda\lambda + 2$ and thus the eigenvalues are $1, 2$. Next we calculate corresponding eigenvectors. One checks these are for example $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Let $P = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$. Then, we have, $A = PDP^{-1}$ where D is the diagonal matrix with $1, 2$ in the main diagonal. Thus, $A^k = PD^kP^{-1}$ and $P^{-1} =$

$$\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}.$$

$$A^k = P \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix}$$
$$= \begin{bmatrix} 4 - 3 \cdot 2^k & 12(2^k - 1) \\ 1 - 2^k & -3 + 4 \cdot 2^k \end{bmatrix}$$

(10) Let

$$A = \begin{bmatrix} 0 & 0.3 \\ -0.3 & 0 \end{bmatrix}$$

Then the map $\mathbf{x} \mapsto A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^2$ is a scaling and rotation. Give the scaling factor and the angle of rotation φ with $-\pi < \varphi \leq \pi$.

The scaling factor is $r = \sqrt{0^2 + (0.3)^2} = 0.3$. Then $\frac{1}{r}A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then we solve for $\cos\varphi = 0, \sin\varphi = -1$. Thus $\varphi = -\pi/2$.

(11) Construct the general **real** solution(s) to the system of linear differential equation $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}$$

We calculate the eigenvalues of A as usual. The characteristic polynomial is $\lambda^2 + 4$ and so the eigenvalues are $\pm 2i$. Next we calculate the eigenvectors. An eigenvector for $2i$ is $\mathbf{v} = \begin{bmatrix} 1 \\ 2 + 2i \end{bmatrix}$ and thus the eigenvector for $-2i$ is the complex conjugate, $\overline{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 - 2i \end{bmatrix}$. Then, a basis for the solutions is,

$$\mathbf{y}_1(t) = (\operatorname{Re}\mathbf{v})\cos 2t - (\operatorname{Im}\mathbf{v}\sin 2t$$
$$\mathbf{y}_2(t) = (\operatorname{Re}\mathbf{v})\sin 2t + (\operatorname{Im}\mathbf{v})\cos 2t$$

See page 317-318.

(12) Decide whether the following are true or false (with justification).

(a) If an $n \times n$ matrix has fewer than n distinct eigenvalues, it is not diagonalizable.

False. Consider the 2×2 identity matrix. It is diagonal, but has only one eigenvalue.

(b) Two eigenvectors corresponding to the same eigenvalue are linearly dependent.
False. Take again 2×2 identity matrix. Then, both $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent, but eigenvectors for the same (and only) eigenvalue 1.

(c) If A is row equivalent to the identity matrix, then it is diagonalizable.
False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. This is row equivalent to the identity matrix (subtract first row from the second). But it is not diagonalizable. To see this, notice that being lower triangular, it has only one eigenvalue, 1. The eigenspace corresponding to 1 is one dimensional with a basis $\mathbf{e}_2$. If it were diagonalizable, the eigenspace must be two dimensional.