MATH 217 – WORKSHEET 12

Q.1 Replace each of the following differential equations by an equivalent system of of first-order equations.

(a) $y'' - xy' - xy = 0$

Solution: Let $y_0 = y$ and $y_1 = y'$. The equivalent system is then

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= xy_1 + xy_0 \end{aligned}$$

(b) $y''' = y'' - x^2(y')^2$

Solution: Let $y_0 = y$, $y_1 = y'$, and $y_2 = y''$. The equivalent system is then

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= y_2 \\ y_2' &= y_2 - x^2 y_1^2 \end{aligned}$$

(c) $y^{(4)} - xy''' + x^2y'' - x^3y = 1$

Solution: Let $y_0 = y$, $y_1 = y'$, $y_2 = y''$, and $y_3 = y'''$. The equivalent system is then

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= xy_3 - x^2y_2 + x^3y_0 + 1 \end{aligned}$$

Q.2 Consider the homogeneous first-order linear system

$$\begin{aligned} x' &= x + 3y \\ y' &= 3x + y \end{aligned}$$

(a) Show that $(x, y) = (e^{4t}, e^{4t})$ and $(x, y) = (e^{-2t}, -e^{-2t})$ are solutions.

Solution: Check by differentiation.

(b) Show in two ways that the given solutions are linearly independent on every closed interval. Then write the general solution of this system.

Solution: Check linear independence from the definition: suppose that

$$A(e^{4t}, e^{4t}) + B(e^{-2t}, -e^{-2t}) = (0, 0).$$

Then $Ae^{4t} + Be^{-2t} = 0$ and $Ae^{4t} - Be^{-2t} = 0$, so dividing by e^{4t} (which is never zero) and rearranging gives

$$A = Be^{-6t} = -Be^{-6t},$$

from which we conclude that $A = -A$, so $A = 0$ and thus $B = 0$.

Check linear independence using the Wronskian:

$$W(t) = \det \begin{pmatrix} e^{4t} & e^{-2t} \\ e^{4t} & -e^{-2t} \end{pmatrix} = -2e^{2t},$$

which is nonzero at every t, hence is nonvanishing on every closed interval.

Apply Theorem 10.2.6 to write the general solution

$$\begin{aligned} x(t) &= Ae^{4t} + Be^{-2t} \\ y(t) &= Ae^{4t} - Be^{-2t} \end{aligned}$$

for some constants A, B.

(c) Find the particular solution for which $x(0) = 5$ and $y(0) = 1$.

Solution: Solve the linear system for A, B:

$$\begin{aligned} 5 = x(0) &= A + B \\ 1 = y(0) &= A - B \end{aligned}$$

to get $A = 3$, $B = 2$, giving $\boxed{x(t) = 3e^{4t} + 2e^{-2t}}$ and $\boxed{y(t) = 3e^{4t} - 2e^{-2t}}$.

Q.3 Consider the first-order linear system

$$\begin{aligned} x' &= x + 2y + t - 1 \\ y' &= 3x + 2y - 5t - 2 \end{aligned}$$

(a) Show that $(x, y) = (2e^{4t}, 3e^{4t})$ and $(x, y) = (e^{-t}, -e^{-t})$ are solutions to the homogeneous part of the equation.

Solution: The homogeneous part is

$$\begin{aligned} x' &= x + 2y \\ y' &= 3x + 2y \end{aligned}$$

Check the solutions by differentiation.

(b) Show that the homogeneous solutions from part (a) are linearly independent on every closed interval.

Solution: Check linear independence using the Wronskian:

$$W(t) = \det \begin{pmatrix} 2e^{4t} & e^{-t} \\ 3e^{4t} & -e^{-t} \end{pmatrix} = -5e^{3t},$$

which is nonzero at every t, hence is nonvanishing on every closed interval.

(c) Show that $x(t) = 3t - 2$ and $y(t) = -2t + 3$ give a particular solution to the full equation. Then write the general solution to the full equation.

Solution: Check by differentiation. Then apply Theorem 10.2.7 to write the general solution

$$\begin{aligned} x(t) &= 2Ae^{4t} + Be^{-t} + 3t - 2 \\ y(t) &= 3Ae^{4t} - Be^{-t} - 2t + 3 \end{aligned}$$

for some constants A, B.

(d) Find the solution to the full equation satisfying $x(0) = 2$ and $y(0) = 4$.

Solution: Solve the linear system for A, B:

$$\begin{aligned} 2 = x(0) &= 2A + B - 2 \\ 4 = y(0) &= 3A - B + 3 \end{aligned}$$

to get $A = 1$ and $B = 2$. This gives the solution $\boxed{x(t) = 2e^{4t} + 2e^{-t} + 3t - 2}$ and $\boxed{y(t) = 3e^{4t} - 2e^{-t} - 2t + 3}$.

Q.4 Find the general solution to each of the following homogeneous linear first order systems with constant coefficients, using the method of eigenvalues:

(a)

$$\begin{aligned} x' &= -3x + 4y \\ y' &= -2x + 3y \end{aligned}$$

Solution: Eigenvalues are distinct reals: $-1, 1$. Corresponding eigenvectors are

$$-1: \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad 1: \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution is therefore

$$\begin{aligned} x(t) &= 2Ae^{-t} + Be^{t} \\ y(t) &= Ae^{-t} + Be^{t} \end{aligned}$$

for arbitrary constants A, B.

(b)

$$\begin{aligned} x' &= 4x - 3y \\ y' &= 8x - 6y \end{aligned}$$

Solution: Eigenvalues are distinct reals: $-1, 0$. Corresponding eigenvectors are

$$-2: \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad 0: \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

General solution is therefore

$$\begin{aligned} x(t) &= Ae^{-2t} + 3B \\ y(t) &= 2Ae^{-t} + 4B \end{aligned}$$

for arbitrary constants A, B.

(c)

$$\begin{aligned} x' &= 4x - 2y \\ y' &= 5x + 2y \end{aligned}$$

Solution: Eigenvalues are complex conjugates: $3 + 3i, 3 - 3i$. Corresponding eigenvectors are

$$3 + 3i: \begin{bmatrix} 1 + 3i \\ 5 \end{bmatrix} \qquad 3 - 3i: \begin{bmatrix} 1 - 3i \\ 5 \end{bmatrix}$$

General solution is therefore

$$\begin{aligned} x(t) &= e^{3t}[A(\cos 3t - 3\sin 3t) + B(\sin 3t + 3\cos 3t)] \\ y(t) &= e^{3t}[A(5\cos 3t) + B(5\sin 3t)] \end{aligned}$$

for arbitrary constants A, B.

(d)

$$\begin{aligned} x' &= 5x + 4y \\ y' &= -x + y \end{aligned}$$

Solution: Eigenvalues are repeated reals: $3, 3$. Corresponding eigenvector and generalized eigenvector (scaled to have integer coordinates) are

$$3: \begin{bmatrix} 10 \\ -5 \end{bmatrix} \qquad 3,3: \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Note: The generalized eigenvector $(v_x, v_y) = (1, 2)$ is a nonzero solution to

$$\begin{pmatrix} 5-3 & 4 \\ -1 & 1-3 \end{pmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

The general solution is therefore

$$\begin{aligned} x(t) &= A(10e^{3t}) + B(10t+1)e^{3t} \\ y(t) &= A(-5e^{3t}) + B(-5t+2)e^{3t} \end{aligned}$$

for arbitrary constants A, B. This may be checked with Macsyma:

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x:10*A*exp(3*t)+B*(10*t+1)*exp(3*t);
y:-5*A*exp(3*t)+B*(-5*t+2)*exp(3*t);
expand(diff(x,t)-5*x-4*y); /* should be 0 */
expand(diff(y,t)+x-y); /* should be 0 */
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Q.5 Consider the Volterra predator-prey equation for $x(t)$ and $y(t)$ on $t \geq 0$:

$$\begin{aligned} x' &= ax - bxy \\ y' &= -cy + gxy, \end{aligned}$$

where a, b, c, g are all positive.

(a) Find the equilibrium solution in terms of a, b, c, d.

Solution: Put $x' = 0$ and $y' = 0$ and solve to get

$$x(t) = \frac{c}{g}, \qquad y(t) = \frac{a}{b},$$

both independent of time.

(b) Show that if $x'(t) > 0$, then $y'' > 0$.

Solution: Differentiate the second Volterra equation (for predator y):

$$y'' = -cy' + gx'y + gxy' = (-c + gx)y' + gx'y = \frac{(y')^2}{y} + gx'y,$$

substituting $-c + gx = y'/y$ at the last step. Observe that if $x' > 0$, then the final expression must be positive.

(c) Give an interpretation of the result from part (b) in the model for prey x and predators y.

Solution: Population growth of predators accelerates when the prey population is increasing.

Q.6 (Optional) Let

$$A = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}, \qquad u(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad u_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},$$

Solve the linear system $u' = Au$ with initial condition $u(0) = u_0$ by finding the matrix exponential e^{tA}.

(Hint: use the eigenvalues and eigenvectors from Q.4(a).)

Solution: The solution will be $u(t) = e^{tA}u_0$. The matrix exponential can be found by diagonalization to the matrix of eigenvalues

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

using the eigenvectors as the columns of a similarity transformation S:

$$S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad S^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}; \qquad A = SDS^{-1}.$$

But then

$$e^{tA} = Se^{tD}S^{-1} = S \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} S^{-1} = \begin{pmatrix} 2e^{-t} - e^{t} & -2e^{-t} + 2e^{t} \\ e^{-t} - e^{t} & -e^{-t} + 2e^{t} \end{pmatrix}$$

so

$$u(t) = e^{tA}u_0 = \begin{bmatrix} (2x_0 - 2y_0)e^{-t} + (-x_0 + 2y_0)e^{t} \\ (x_0 - y_0)e^{-t} + (-x_0 + 2y_0)e^{t} \end{bmatrix}$$

Remark. Put $A = x_0 - y_0$ and $B = -x_0 + 2y_0$ to get the Q.4(a) solution.