MATH 217 – WORKSHEET 11

Q.1 In parts (a) and (b) below, constant $a > 0$ and function $f = f(x)$ are given, and $w = w(x,t)$ is to be found.

(a) Solve the boundary value problem

$$\begin{aligned} a^2 w_{xx} &= w_t, \qquad 0 < x < \pi,\ t \geq 0, \\ w(x,0) &= f(x), \quad 0 < x < \pi, \\ w(0,t) &= 0, \quad t \geq 0, \\ w(\pi,t) &= 0, \quad t \geq 0. \end{aligned}$$

Solution: Following Section 11.3, we get the solution

$$w(x,t) = \sum_{j=1}^{\infty} b_j e^{-j^2 a^2 t} \sin jx$$

which, for all sequences $\{b_j\}$, satisfies $w(0,t) = w(\pi,t) = 0$ for all $t \geq 0$. Then we determine $\{b_j : j = 1, 2, \dots\}$ from the Fourier sine series for f:

$$b_j = \frac{2}{\pi} \int_0^{\pi} f(x) \sin jx \, dx.$$

(b) Solve the boundary value problem in part (a) with the modified end conditions

$$w(0,t) = w_1, \qquad w(\pi,t) = w_2, \quad t \geq 0,$$

where w_1 and w_2 are given real numbers.

Solution: Let

$$g(x,t) = \frac{(w_2 - w_1)}{\pi} x + w_1$$

This is chosen so that $g(0,t) = w_1$ and $g(\pi,t) = w_2$ for all t. Note that $g_{xx} = 0$ and $g_t = 0$ for all x, t, so g is a solution to the PDE with the modified end conditions.

Now suppose that w solves the initial-boundary value problem of part (a), only with initial shape $f(x)$ replaced by $f(x) - g(x,0)$. Then

$$w(x,t) = \sum_{j=1}^{\infty} c_j e^{-j^2 a^2 t} \sin jx$$

with

$$c_j = \frac{2}{\pi}\int_0^\pi [f(x) - g(x,0)]\sin jx\,dx = b_j - \frac{2}{\pi}\int_0^\pi g(x,0)\sin jx\,dx.$$

Then $w(x,t)+g(x,t)$ satisfies the PDE, the modified end conditions, and the original initial shape condition.

Q.2 The two-dimensional heat equation is

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial w}{\partial t}$$

for $w = w(x,y,t)$. Use the method of separation of variables to find the steady-state solution in the infinite half-strip of the xy plane consisting of $\{(x,y) : 0 \le x \le \pi, y \ge 0\}$ if the following boundary conditions are satisfied:

$$\begin{aligned} w(0,y,t) = w(\pi,y,t) &= 0, \quad t \ge 0, y \ge 0 \\ w(x,0,0) &= f(x), \quad 0 \le x \le \pi \\ \lim_{y\to\infty} w(x,y,t) &= 0, \quad t \ge 0, 0 \le x \le \pi. \end{aligned}$$

Solution: Equilibrium solution has $\partial w/\partial t = 0$, so there is no t-dependence, and the PDE reduces to

$$a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial w}{\partial t} = 0,$$

so a may be canceled giving Dirichlet's equation:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

Now separate variables $w(x,y) = u(x)v(y)$ to get two eigenvalue problems from the PDE:

$$u'' = -\lambda u, \quad u(0) = u(\pi) = 0; \qquad v'' = \lambda v, \quad v(y) \to 0 \text{ as } y \to \infty.$$

The first implies that $\lambda = n^2$ for some integer n and that $u(x) = \sin nx$. The second implies that $v(y) = e^{-ny}$ and that $n > 0$. Combining these gives

$$w(x,y) = \sum_{n=1}^{\infty} b_n e^{-ny}\sin nx,$$

where b_n is the Fourier sine coefficient of the boundary function $f(x)$:

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx,$$

found by setting $y = 0$ to get $w(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = f(x)$.

Q.3 Solve the Dirichlet problem for the unit disc, using the Poisson kernel, when the boundary function $f(\theta)$ is defined by

(a) $f(\theta) = \cos(\theta/2)$, $-\pi \le \theta \le \pi$.

Solution: Use the Poisson kernel method:

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) \cos(\xi/2)}{1 - 2r \cos(\theta - \xi) + r^2} \, d\xi$$

(b) $f(\theta) = 0$ if $-\pi \le \theta < 0$ and $f(\theta) = 1$ if $0 \le \theta \le \pi$.

Solution: Use the Poisson kernel method:

$$u(r, \theta) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \xi) + r^2} \, d\xi$$

(c) $f(\theta) = 0$ if $-\pi \le \theta < 0$ and $f(\theta) = \sin\theta$ if $0 \le \theta \le \pi$.

Solution: Use the Poisson kernel method:

$$u(r, \theta) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{(1 - r^2) \sin(\xi)}{1 - 2r \cos(\theta - \xi) + r^2} \, d\xi$$

Q.4 Given the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

there is an exact equation obtained by multiplying with an integrating factor $\mu(x)$:

$$0 = \mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = [\mu(x)P(x)y']' + [S(x)y]',$$

for some appropriate function $S = S(x)$.

(a) Show that $\mu(x)$ satisfies the *adjoint equation*

$$P(x)\mu'' + [2P'(x) - Q(x)]\mu' + [P''(x) - Q'(x) + R(x)]\mu = 0.$$

Solution: By the product rule:

$$\begin{aligned}(\mu P y')' &= \mu' P y' + \mu P' y' + \mu P y'' \\ (Sy)' &= S' y + S y'\end{aligned}$$

Adding and rearranging gives

$$(\mu P y')' + (Sy)'' = \mu P y'' + (\mu' P + \mu P' + S) y' + S' y.$$

This gives the μ-multiplied exact equation if we set $S' = \mu R$ and

$$\mu' P + \mu P' + S = \mu Q.$$

Differentiating this last equation by the product rule gives

$$\mu'' P + \mu' P' + \mu' P' + \mu P'' + S' = \mu' Q + \mu Q',$$

then substituting for S' and rearranging gives

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0,$$

as claimed.

(b) Find the adjoint equation for Legendre's equation

$$(1 - x^2)y'' - 2xy' + p(p-1)y = 0.$$

Solution: Here $P = 1 - x^2$, $Q = -2x$, and $R = p(p-1)$, so

$$\begin{aligned}2P' - Q &= (2)(-2x) - (-2x) = -2x, \\ P'' - Q' + R &= (-2) - (-2) + p(1-p) = p(1-p),\end{aligned}$$

so the adjoint equation is

$$\boxed{(1 - x^2)\mu'' - 2x\mu' + p(1-p)\mu = 0,}$$

which is again Legendre's equation but for μ.

(c) Find the adjoint equation for Bessel's equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0.$$

Solution: Here $P = x^2$, $Q = x$, and $R = x^2 - p^2$, so

$$\begin{aligned} 2P' - Q &= (2)(2x) - (x) = 3x, \\ P'' - Q' + R &= (2) - (1) + (x^2 - p^2) = x^2 - p^2 + 1, \end{aligned}$$

so the adjoint equation is

$$\boxed{x^2\mu'' + 3x\mu' + (x^2 - p^2 + 1)\mu = 0.}$$

(d) Find the adjoint equation for Hermite's equation

$$y'' - 2xy' + 2py = 0.$$

Solution: Here $P = 1$, $Q = -2x$, and $R = 2p$, so

$$\begin{aligned} 2P' - Q &= (2)(0) - (-2x) = 2x, \\ P'' - Q' + R &= (0) - (-2) + (2p) = 2p + 2, \end{aligned}$$

so the adjoint equation is

$$\boxed{\mu'' + 2x\mu' + (2p + 2)\mu = 0.}$$

(d) Find the adjoint equation for Laguerre's equation

$$xy'' + (1 - x)y' + py = 0.$$

Solution: Here $P = x$, $Q = 1 - x$, and $R = p$, so

$$\begin{aligned} 2P' - Q &= (2)(1) - (1 - x) = 1 + x, \\ P'' - Q' + R &= (0) - (-1) + (p) = p + 1, \end{aligned}$$

so the adjoint equation is

$$\boxed{x\mu'' + (1 + x)\mu' + (p + 1)\mu = 0.}$$

Q.5 The equation $P(x)y'' + Q(x)y' + R(x)y = 0$ is called *self-adjoint* if its adjoint is just the same equation, possibly after a change of notation.

(a) Show that this equation is self-adjoint if and only if $P' = Q$.

Solution: Self-adjointness holds if and only iff the coefficient functions are the same. From the previous problem, we see that the adjoint equation is the same as the original equation if and only if

$$P = P, \qquad 2P' - Q = Q, \qquad P'' - Q' + R = R.$$

The first equality always holds. The second and third hold if and only if $P' = Q$, as claimed.

(b) Show that such a self-adjoint equation may be written as

$$P(x)y'' + P'(x)y' + R(x)y = 0, \quad \Longleftrightarrow \; [P(x)y']' + R(x)y = 0$$

(Hint: use part (a).)

Solution: By part (a), a self-adjoint equation $Py'' + Qy' + Ry = 0$ has $P' = Q$ and may thus be written $Py'' + P'y + Ry = 0$. Apply the product rule $[Py']' = Py'' + P'y'$ to obtain the claimed result.

(c) Determine which of the equations in Q.4 above are self-adjoint.

Solution: Only Legendre's equation satisfies $P' = Q$, and only Legendre's equation is self-adjoint.