MATH 217 – WORKSHEET 10

Q.1 Let δ be the Dirac mass at 0, so $\delta(x-a)$ is the Dirac mass at a. Let h be the Heaviside function with a jump at $x=0$, so $h(x-a)$ is the Heaviside function with a jump at $x=a$.

(a) Find

$$L\left[\sum_{n=0}^{\infty}\delta(x-n)\right](p)$$

Solution: Expand the sum and use $L[\delta(x-a)](p)=e^{-ap}$, then apply the geometric sum formula:

$$\sum_{n=0}^{\infty}L\left[\delta(x-n)\right](p)=\sum_{n=0}^{\infty}e^{-np}=\sum_{n=0}^{\infty}(e^{-p})^n=\frac{1}{1-e^{-p}}=\boxed{\frac{e^p}{e^p-1}.}$$

(b) Find

$$L\left[\sum_{n=0}^{\infty}h(x-n)\right](p)$$

Solution: Expand the sum and use $L[h(x-a)](p)=e^{-ap}/p$, then apply the geometric sum formula:

$$\sum_{n=0}^{\infty}L\left[h(x-n)\right](p)=\frac{1}{p}\sum_{n=0}^{\infty}e^{-np}=\sum_{n=0}^{\infty}(e^{-p})^n=\frac{1}{p(1-e^{-p})}=\boxed{\frac{e^p}{p(e^p-1)}.}$$

(c) Find $L[s(x)](p)$, where $s:[0,\infty)\to\mathbf{R}$ is the square wave function

$$s(x)=\begin{cases}1, & 2k<x<2k+1,\\ -1, & 2k+1<x<2k+2,\end{cases}\qquad k=0,1,2,\ldots$$

Solution: First write s in terms of Heaviside functions:

$$s(x)=h(x)+2\sum_{n=1}^{\infty}(-1)^n h(x-n)$$

Expand the sum and use $L[h(x-a)](p)=e^{-ap}/p$, then apply the geometric sum formula:

$$L[s(x)](p)=\frac{1}{p}+\frac{2}{p}\sum_{n=1}^{\infty}(-1)^n e^{-np}=\frac{1}{p}+\frac{2}{p}\sum_{n=0}^{\infty}(-e^{-p})^n=\boxed{\frac{1-e^{-p}}{p(1+e^{-p})}.}$$

(d) Find $L[|\sin x|](p)$, the Laplace transform of the "full wave rectified" sine wave $|\sin x|$. (Hint: use periodicity.)

Solution: Note that $|\sin x|$ is π-periodic, so the Laplace transform may be expanded in a series:

$$\begin{aligned} L[|\sin x|](p) &= \int_0^\infty |\sin x| e^{-px}\, dx \\ &= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} |\sin x| e^{-px}\, dx \\ &= \sum_{n=0}^\infty \int_0^\pi |\sin(x+n\pi)| e^{-p(x+n\pi)}\, dx \\ &= \sum_{n=0}^\infty e^{-pn\pi} \int_0^\pi \sin(x) e^{-px}\, dx \\ &= \boxed{\left(\frac{1}{p^2+1}\right)\frac{1+e^{-\pi p}}{1-e^{-\pi p}}.} \end{aligned}$$

Here we used integration by parts twice to obtain

$$\int_0^\pi e^{-px}\sin x\, dx = \frac{1+e^{-\pi p}}{p^2+1},$$

which has no n-dependence and thus factors out of the sum, and then we applied the geometric sum formula:

$$\sum_{n=0}^\infty e^{-pn\pi} = \sum_{n=0}^\infty (e^{-p\pi})^n = \frac{1}{1-e^{-\pi p}}.$$

Q.2 Find the eigenvalues λ_n and the eigenfunctions y_n for the equation $y'' + \lambda y = 0$ in each of the following instances:

(a) $y(0) = 0$, $y(\pi/2) = 0$.

Solution: Basic solutions are $\sin\sqrt{\lambda}\,x$ and $\cos\sqrt{\lambda}\,x$. Sines and only sines satisfy $y(0) = 0$. Eigenvalues must also satisfy

$$0 = y(\pi/2) = \sin(\sqrt{\lambda}\frac{\pi}{2}) \implies \lambda \in \{2^2, 4^2, 6^2, \dots\}.$$

(Zero is not an eigenvalue because there is no nonzero eigenfunction: only $y \equiv 0$ satisfies $y'' + 0y = 0$ and the boundary conditions.) Conclude that $\{\lambda_n =$

$(2n)^2 : n = 1, 2, \dots\}$ are the eigenvalues, with corresponding eigenfunctions $\{\sin\sqrt{\lambda_n}\, x = \sin 2nx : n = 1, 2, \dots\}$.

(b) $y(0) = 0$, $y(2\pi) = 0$.

Solution: Eigenvalues $\{\lambda_n = (n/2)^2 : n = 1, 2, \dots\}$. Corresponding eigenfunctions $\{\sin\sqrt{\lambda_n}\, x = \sin\frac{n}{2}x : n = 1, 2, \dots\}$.

(c) $y(0) = 0$, $y(1) = 0$.

Solution: Eigenvalues $\{\lambda_n = (n\pi)^2 : n = 1, 2, \dots\}$. Corresponding eigenfunctions $\{\sin\sqrt{\lambda_n}\, x = \sin n\pi x : n = 1, 2, \dots\}$.

(d) $y(a) = 0$, $y(b) = 0$ for fixed $a < b$.

Solution: Eigenvalues $\{\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 : n = 1, 2, \dots\}$. Corresponding eigenfunctions $\{\sin\sqrt{\lambda_n}(x-a) = \sin\frac{n\pi(x-a)}{b-a} : n = 1, 2, \dots\}$.

Q.3 Let F, G be arbitrary twice-differentiable functions of one variable. Fix $a > 0$ and let $y(x,t) = F(x+at) + G(x-at)$.

(a) Show that y satisfies the partial differential equation

$$a^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}.$$

Solution: By the chain rule:

$$\frac{\partial^2 F(x+at)}{\partial x^2} = F''(x+at); \quad \frac{\partial^2 F(x+at)}{\partial t^2} = a^2 F''(x+at).$$

Likewise,

$$\frac{\partial^2 G(x-at)}{\partial x^2} = G''(x-at); \quad \frac{\partial^2 G(x-at)}{\partial t^2} = a^2 G''(x-at).$$

Thus $F(x+at)$ and $G(x-at)$ are solutions. Since the PDE is linear, their sum is also a solution.

(b) Let $\alpha = x + at$ and $\beta = x - at$. Show that, with this change of variables for y, the equation from part (a) may be written as

$$\frac{\partial^2 y}{\partial\alpha\partial\beta} = 0.$$

Solution: Solve for $x = (\alpha+\beta)/2$ and $t = (\alpha-\beta)/(2a)$ to get

$$y(x,t) = y\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2a}\right),$$

where $y_x = \partial y/\partial t$ and $y_t = \partial y/\partial t$. By the chain rule,

$$\frac{\partial y}{\partial \beta} = \frac{1}{2} y_x \left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2a} \right) - \frac{1}{2a} y_t \left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2a} \right)$$

and thus

$$\frac{\partial^2 y}{\partial \alpha \partial \beta} = \frac{1}{4} y_{xx} + \frac{1}{4a} y_{xt} - \frac{1}{4a} y_{tx} - \frac{1}{4a^2} y_{tt} = \frac{1}{4a^2} \left(a^2 y_{xx} - y_{tt} \right),$$

using the equality of mixed partial derivatives $y_{xt} = y_{tx}$ to cancel.

(c) From the equation in part (b), integrate twice to derive the formula for y in terms of F and G. (Note: this is known as d'Alembert's solution of the wave equation.)

Solution: Integrating with respect to α gives

$$\frac{\partial y}{\partial \beta} = g(\beta),$$

where g is an arbitrary differentiable function of β alone so that $\partial g/\partial \alpha = 0$. Integrating again with respect to β gives

$$y = G(\beta) + F(\alpha),$$

where $G'(\beta) = g(\beta)$ and $F(\alpha)$ is an arbitrary function of α alone.

Q.4 Consider the vibrating string problem in the text (Eqs.11.9, 11.10, and 11.11 on p.322).

(a) Find the solution if the initial shape is $y(x,0) = c \sin x$ on $x \in [0, \pi]$.

Solution: All solutions are of the form

$$y = \alpha_1 \sin x \cos t + \cdots + \alpha_n \sin nx \cos nt + \cdots = \sum_{n=1}^{\infty} \alpha_n \sin nx \cos nt.$$

The initial shape condition $y(x,0) = c \sin x$ implies that

$$c \sin x = \sum_{n=1}^{\infty} \alpha_n \sin nx$$

from which we conclude that $\alpha_1 = c$ but $\alpha_n = 0$ for $n = 2, 3, \ldots$. Thus $\boxed{y(x,t) = c \sin x \cos t.}$

(b) Find the solution if the initial shape is $y(x,0) = c\sin nx$ on $x \in [0,\pi]$, for some positive integer n.

Solution: The reasoning in part (a) leads to the solution $\boxed{y(x,t) = c\sin nx\cos nt.}$

Remark. The solution in part (b) factors into a function of x times a function of y called a "standing wave." Such a standing wave solution has $n+1$ equispaced values $0 = x_0 < \cdots < x_n = \pi$ call "nodal points" where $y(x_k,t) = 0$ for all t.

Q.5 Solve the "struck string" problem, namely the wave equation $a^2 y_{xx} = y_{tt}$ with boundary conditions $y(0,t) = y(\pi,t) = 0$, all $t \geq 0$, and the initial conditions

$$y(x,0) = 0; \qquad y_t(x,0) = g(x),$$

for $x \in [0,\pi]$, with a given function $g = g(x)$ defined on $[0,\pi]$.

Solution: Much like Eq.11.15 on p.323 of the textbook, the solution is given by

$$\boxed{y(x,t) = \sum_{j=1}^{\infty} b_j \sin jx \sin jat,}$$

which satisfies the PDE and the boundary conditions. The set $\{b_j : j = 1,2,\dots\}$ lists the Fourier sine coefficients for the initial velocity function $g(x)$, which must satisfy

$$g(x) = \frac{\partial y}{\partial t}\Big|_{t=0} = \sum_{j=1}^{\infty} b_j \sin jx(ja\cos jat)|_{t=0} = \sum_{j=1}^{\infty} jab_j \sin jx.$$

Compute these coefficients using the formula in Ch.7.3, p.180 of the textbook:

$$jab_j = \frac{2}{\pi}\int_0^{\pi} g(x)\sin jx\,dx \qquad \Longrightarrow \boxed{b_j = \frac{2}{\pi ja}\int_0^{\pi} g(x)\sin jx\,dx.}$$