MATH 217 – WORKSHEET 09

Q.1 Solve each of the following initial value problems using the Laplace transform.

(a) $y' + y = e^{2x}$, $y(0) = 0$.

Solution: Transforming both sides gives $L[y' + y](p) = L[e^{2x}](p)$, which evaluates to

$$(p+1)L[y](p) - y(0) = \frac{1}{p-2}.$$

But then

$$L[y](p) = \frac{1}{(p+1)(p+2)} = \frac{1}{p^2 - p - 2} = \frac{1}{(p-\frac{1}{2})^2 - \frac{9}{4}},$$

(using $y(0) = 0$, dividing by $p+1$, and completing the squares), which gives

$$\boxed{y(x) = L^{-1}\left[\frac{1}{(p-\frac{1}{2})^2 - \frac{9}{4}}\right](x) = \frac{2}{3}\sinh(\frac{3}{2}x)e^{x/2} = \frac{1}{3}e^{2x} - \frac{1}{3}e^{-x}}$$

by partial fractions or by Eq.8.4 and table lookup.

(b) $y'' + 2y' + 2y = 2$, $y(0) = 0$ and $y'(0) = 1$.

Solution: Transforming both sides gives $L[y'' + 2y' + 2y](p) = L[2](p)$, which evaluates to

$$(p^2 + 2p + 2)L[y](p) - (p+1)y(0) - y'(0) = \frac{2}{p}.$$

But then

$$L[y](p) = \frac{1 + 2/p}{p^2 + 2p + 2} = \frac{p+2}{p([p+1]^2 + 1^2)} = \frac{1}{p} - \frac{[p+1]}{[p+1]^2 + 1^2}$$

(using $y(0) = 0$ and $y'(0) = 1$, dividing by $p^2 + 2p + 2 = [p+1]^2 + 1^2$, and partial fraction decomposition), which gives

$$\boxed{y(x) = L^{-1}\left[\frac{1}{p} - \frac{[p+1]}{[p+1]^2 + 1^2}\right](x) = 1 - e^{-x}\cos x}$$

by Eq.8.4 and table lookup.

(c) $y'' + y' = 3x^2$, $y(0) = 0$ and $y'(0) = 1$

Solution: Transforming both sides gives $L[y''+y'](p) = L[3x^2](p)$, which evaluates to

$$(p^2 + p)L[y](p) - py(0) - y'(0) = \frac{6}{p^3}.$$

But then

$$L[y](p) = \frac{1 + 9/p^3}{p^2 + p} = \frac{p^3 + 6}{p^4(p + 1)} = \frac{5}{p+1} - \frac{5}{p} + \frac{6}{p^2} - \frac{6}{p^3} + \frac{6}{p^4},$$

(using $y(0) = 0$ and $y'(0) = 1$, dividing by $p^2 + p = p(p + 1)$, and partial fraction decomposition), which gives

$$\boxed{y(x) = L^{-1}\left[\frac{5}{p+1} - \frac{5}{p} + \frac{6}{p^2} - \frac{6}{p^3} + \frac{6}{p^4}\right](x) = 5e^{-x} - 5 + 6x - 3x^2 + x^3}$$

by Eq.8.4 and table lookup.

Remark. Macsyma can find partial fraction expansions and then check them, as follows:

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partfrac((p^3+6)/(p^4*(p+1)),p);
ratsimp(6/p^4-6/p^3+6/p^2-5/p+5/(p+1));
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Q.2 Use the formula $L[y'(x)](p) = pL[y(x)](p) - y(0)$ to derive the formula

$$L\left[\int_0^x f(t)\,dt\right](p) = \frac{1}{p}L[f(x)](p).$$

Solution: Let y be an antiderivative for f, so $y'(x) = f(x)$, so

$$L[f(x)](p) = L[y'(x)](p) = pL[y(x)](p) - y(0).$$

By the Fundamental Theorem of Calculus,

$$\int_0^x f(t)\,dt = y(x) - y(0),$$

so using $L[y(0)](p) = y(0)/p$ and the first identity above,

$$L\left[\int_0^x f(t)\,dt\right](p) = L[y(x)](p) - \frac{1}{p}y(0) = \frac{1}{p}\left(pL[y(x)](p) - y(0)\right) = \frac{1}{p}L[f(x)](p),$$

as claimed.

Q.3 Without worrying about convergence issues, show the following:

(a) $\displaystyle\int_0^\infty \frac{\sin yx}{x}\,dx = \frac{\pi}{2}$, for all $y > 0$

Solution: Let $f(x) = \sin yx$, so $F(p) = L[f(x)](p) = y/(p^2+y^2)$. Then by Eq.8.12,

$$\int_0^\infty \frac{\sin yx}{x}\,dx = L\left[\frac{f(x)}{x}\right](0) = L\left[\frac{\sin yx}{x}\right](0) = \int_{s=0}^\infty F(s)\,ds.$$

Now substitute $s = yu$, which for any $y > 0$ gives

$$\int_{s=0}^\infty F(s)\,ds = \int_{s=0}^\infty \frac{y}{s^2+y^2}\,ds = \int_{u=0}^\infty \frac{1}{u^2+1}\,du = \arctan u\Big|_{u=0}^\infty = \frac{\pi}{2}.$$

This, combined with the prior computation, gives the result.

(b) $\displaystyle\int_0^\infty \frac{\cos yx}{1+x^2}\,dx = \frac{\pi}{2}e^{-y}$, for all $y > 0$.

Solution: Let $f(y) = \displaystyle\int_0^\infty \frac{\cos yx}{1+x^2}\,dx$. Then by twice differentiating under the integral, obtain

$$f''(y) = \int_0^\infty \frac{-x^2\cos yx}{1+x^2}\,dx.$$

(This integral may not converge, but we are not supposed to worry about that.) Thus,

$$f(y) - f''(y) = \int_0^\infty \cos yx\,dx = L[\cos yx](0) = \frac{p}{p^2+y^2}\Big|_{p=0} = 0$$

This homogeneous ODE has the general solution $f(y) = Ae^y + Be^{-y}$. It remains to determine A and B. But

$$A + B = f(0) = \int_0^\infty \frac{\cos 0}{1+x^2}\,dx = \arctan x\Big|_0^\infty = \frac{\pi}{2},$$

while

$$\lim_{y\to\infty} f(y) = \lim_{y\to\infty} \int_0^\infty \frac{\cos yx}{1+x^2}\,dx = 0$$

by the Riemann-Lebesgue Lemma on p.197 of the textbook. (In fact it suffices to observe that $f(y)$ is bounded for all $y > 0$.) Conclude from this that $A = 0$, and thus $B = \pi/2$, so $\boxed{f(y) = \frac{\pi}{2}e^{-y}}$

Q.4 Compute the convolution $f*g(x)$ of the following pairs of functions:

(a) $f(x) = e^{ax}$, $g(x) = e^{bx}$.

Solution:

$$\int_0^x e^{at}e^{b(x-t)}\,dt = e^{bx}\int_0^x e^{(a-b)t}\,dt = \frac{e^{bx}}{a-b}[e^{(a-b)x}-1] = \frac{e^{ax}-e^{bx}}{a-b}$$

(b) $f(x) = e^x$, $g(x) = x$.

Solution:

$$\int_0^x (x-t)e^t\,dt = x\int_0^x e^t\,dt - \int_0^x te^t\,dt = x(e^x-1)-[xe^x-(e^x-1)] = e^x-x-1,$$

using integration by parts in the second integral.

(c) $f(x) = \sin at$, $g(x) = \sin bt$, where $a^2 \neq b^2$.

Solution: Use the identity

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

and the hypothesis that $a^2 \neq b^2$ (so $a \pm b \neq 0$) to compute

$$\begin{aligned}
\int_0^x \sin at \sin b(x-t)\,dt &= \frac{1}{2}\int_0^x \Big[\cos[(a+b)t-bx] - \cos[(a-b)t+bx]\Big]\,dt \\
&= \frac{1}{2}\left[\frac{\sin[(a+b)t-bx]}{a+b} - \frac{\sin[(a-b)t+bx]}{a-b}\right]_0^x \\
&= \frac{1}{2}\left[\frac{\sin ax+\sin bx}{a+b} - \frac{\sin ax-\sin bx}{a-b}\right] \\
&= \frac{a\sin bx - b\sin ax}{a^2-b^2}
\end{aligned}$$

Q.5 For $a \geq 0$, let h_a be the unit step function with a jump at a:

$$h_a(x) = \begin{cases} 0, & x < a, \\ 1, & x \geq a. \end{cases}$$

(a) Compute the Laplace transform of h_a.

Solution: From the definition, at any $p > 0$,

$$L[h_a(x)](p) = \int_a^\infty e^{-px}\,dx = \frac{e^{-px}}{-p}\Big|_a^\infty = \frac{1}{p}e^{-pa}.$$

(b) Compute the convolution $h_a * h_b(x)$ for $a \geq 0$ and $b \geq 0$.

Solution: First note that

$$h_a(t)h_b(x-t) = \begin{cases} 1, & t \geq a \text{ and } x - t \geq b, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 1, & a \leq t \leq x - b, \\ 0, & \text{otherwise,} \end{cases}$$

So by the definition of convolution,

$$h_a * h_b(x) = \int_0^x h_a(t)h_b(x-t)\,dt = \begin{cases} x - (a+b), & x > a + b, \\ 0, & \text{otherwise.} \end{cases}$$

One way to write this function is

$$\boxed{h_a * h_b(x) = [x - (a+b)]h_{a+b}(x)}.$$

(c) How could you compute and interpret $L[h_a'(x)]$ for $a > 0$?

Solution: Using the derivative formula $L[y'](p) = pL[y](p) - y(0)$ gives

$$L[h_a'(x)](p) = p\frac{1}{p}e^{-pa} - h_a(0) = e^{-pa}.$$

This suggests that $h_a'(x)$ behaves like a unit impulse at $x = a$.

Worksheet 10 Q.1 Let δ be the Dirac mass at 0, so $\delta(x-a)$ is the Dirac mass at a. Let h be the Heaviside function with a jump at $x = 0$, so $h(x-a)$ is the Heaviside function with a jump at $x = a$.

(a) Find

$$L\left[\sum_{n=0}^{\infty} \delta(x-n)\right](p)$$

Solution: Expand the sum and use $L[\delta(x-a)](p) = e^{-ap}$, then apply the geometric sum formula:

$$\sum_{n=0}^{\infty} L\left[\delta(x-n)\right](p) = \sum_{n=0}^{\infty} e^{-np} = \sum_{n=0}^{\infty} (e^{-p})^n = \frac{1}{1-e^{-p}} = \boxed{\frac{e^p}{e^p - 1}.}$$

(b) Find

$$L\left[\sum_{n=0}^{\infty} h(x-n)\right](p)$$

Solution: Expand the sum and use $L[h(x-a)](p) = e^{-ap}/p$, then apply the geometric sum formula:

$$\sum_{n=0}^{\infty} L\left[h(x-n)\right](p) = \frac{1}{p}\sum_{n=0}^{\infty} e^{-np} = \sum_{n=0}^{\infty}(e^{-p})^n = \frac{1}{p(1-e^{-p})} = \boxed{\frac{e^p}{p(e^p-1)}.}$$

(c) Find $L[s(x)](p)$, where $s : [0,\infty) \to \mathbf{R}$ is the square wave function

$$s(x) = \begin{cases} 1, & 2k < x < 2k+1, \\ -1, & 2k+1 < x < 2k+2, \end{cases} \qquad k = 0, 1, 2, \ldots$$

Solution: First write s in terms of Heaviside functions:

$$s(x) = h(x) + 2\sum_{n=1}^{\infty}(-1)^n h(x-n)$$

Expand the sum and use $L[h(x-a)](p) = e^{-ap}/p$, then apply the geometric sum formula:

$$L[s(x)](p) = \frac{1}{p} + \frac{2}{p}\sum_{n=1}^{\infty}(-1)^n e^{-np} = \frac{1}{p} + \frac{2}{p}\sum_{n=0}^{\infty}(-e^{-p})^n = \boxed{\frac{1-e^{-p}}{p(1+e^{-p})}.}$$

(d) Find $L[|\sin x|](p)$, the Laplace transform of the "full wave rectified" sine wave $|\sin x|$. (Hint: use periodicity.)

Solution: Note that $|\sin x|$ is π-periodic, so the Laplace transform may

be expanded in a series:

$$\begin{aligned} L[|\sin x|](p) &= \int_0^\infty |\sin x| e^{-px}\, dx \\ &= \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} |\sin x| e^{-px}\, dx \\ &= \sum_{n=0}^\infty \int_0^\pi |\sin(x+n\pi)| e^{-p(x+n\pi)}\, dx \\ &= \sum_{n=0}^\infty e^{-pn\pi} \int_0^\pi \sin(x) e^{-px}\, dx \\ &= \boxed{\left(\frac{1}{p^2+1}\right)\frac{1+e^{-\pi p}}{1-e^{-\pi p}}.} \end{aligned}$$

Here we used integration by parts twice to obtain

$$\int_0^\pi e^{-px}\sin x\, dx = \frac{1+e^{-\pi p}}{p^2+1},$$

which has no n-dependence and thus factors out of the sum, and then we applied the geometric sum formula:

$$\sum_{n=0}^\infty e^{-pn\pi} = \sum_{n=0}^\infty (e^{-p\pi})^n = \frac{1}{1-e^{-\pi p}}.$$