MATH 217 – WORKSHEET 07

Q.1 This exercise illustrates how to compute Fourier series using trigonometric identities.

(a) Use the double angle formulas $\cos 2x = 1 - 2\sin^2 x = \cos^2 x - 1$ to compute the Fourier series of $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$ without any integration.

Solution: Solve for $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ to conclude that $a_0 = 1$, $a_2 = -\frac{1}{2}$, and all other Fourier coefficients a_j, b_j are zero.

Solve for $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ to conclude that $a_0 = 1$, $a_2 = \frac{1}{2}$, and all other Fourier coefficients a_j, b_j are zero.

(b) Use Euler's formula $e^{ix} = \cos x + i \sin x$ to prove that

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x, \qquad \cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x,$$

and use these to find the Fourier series for $\sin^3 x$ and $\cos^3 x$ without integration.

Solution: First compute $(e^{ix})^3 = e^{i3x} = \cos 3x + i \sin 3x$. But also,

$$\begin{aligned}
(e^{ix})^3 &= (\cos x + i \sin x)^3 \\
&= \cos^3 x + 3i \cos^2 x \sin x + 3(i^2) \cos x \sin^2 x + i^3 \sin^3 x \\
&= \cos^3 x - 3 \cos x \sin^2 x + i \left[3 \cos^2 x \sin x - \sin^3 x\right] \\
&= \cos^3 x - 3 \cos x (1 - \cos^2 x) + i \left[3(1 - \sin^2 x) \sin x - \sin^3 x\right] \\
&= 4 \cos^3 x - 3 \cos x + i \left[3 \sin x - 4 \sin^3 x\right].
\end{aligned}$$

Equate the real and imaginary parts to conclude that

$$4\cos^3 x - 3\cos x = \cos 3x, \qquad \sin x - 4 \sin^3 x = \sin 3x,$$

and then solve for $\cos^3 x$ and $\sin^3 x$, respectively, to derive the claimed identities.

Conclude that the Fourer coefficients of $\cos^3 x$ are $a_1 = \frac{3}{4}$, $a_3 = \frac{1}{4}$, with all other a_j, b_j vanishing.

Likewise, conclude that the Fourer coefficients of $\sin^3 x$ are $b_1 = \frac{3}{4}$, $b_3 = -\frac{1}{4}$, with all other a_j, b_j vanishing.

Q.2 Find the Fourier series for the functions on the given intervals:

(a) $f(x) = x$, $[-1, 1]$.

Solution: Put $L = 1$ and use the formulas on p.186 to compute

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \int_{-1}^{1} x \cos n\pi x \, dx = 0$$

for all $n = 0, 1, 2, \ldots$ because the integrand is an odd function and $[-1, 1]$ is a symmetric interval. Likewise, compute for all $n = 1, 2, \ldots$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = \int_{-1}^{1} x \sin n\pi x \, dx = \frac{2(\sin n\pi - n\pi \cos n\pi)}{n^2\pi^2} = \frac{2(-1)^{n-1}}{n\pi},$$

(b) $g(x) = \cos 2x$, $[-\pi/3, \pi/3]$.

Solution: Put $L = \pi/3$ and use the formulas on p.186 to compute

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \int_{-\pi/3}^{\pi/3}\cos 2x\sin 3nx\,dx = 0,$$

for all $n = 1, 2, \ldots$ because the integrand is an odd function and the interval of integration is symmetric.

Next, compute

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{3}{\pi}\int_{-\pi/3}^{\pi/3}\cos 2x\,dx = \frac{3}{2\pi}\sin 2x\Big|_{-\pi/3}^{\pi/3} = \frac{3\sqrt{3}}{2\pi}.$$

Likewise, compute for all $n = 1, 2, \ldots$

$$\begin{aligned}
a_n &= \frac{3}{\pi}\int_{-\pi/3}^{\pi/3}\cos 2x\cos 3nx\,dx \\
&= \frac{3}{\pi}\int_{-\pi/3}^{\pi/3}\frac{1}{2}[\cos(2+3n)x + \cos(2-3n)x]\,dx \\
&= \frac{3}{2\pi}\left[\frac{\sin(2+3n)x}{2+3n} + \frac{\sin(2-3n)x}{2+3n}\right]_{-\pi/3}^{\pi/3} \\
&= \frac{3}{\pi}\sin\left(\frac{2\pi}{3}\right)(-1)^n\left[\frac{1}{2-3n} + \frac{1}{2+3n}\right] \\
&= \frac{3\sqrt{3}}{2\pi}\left[\frac{(-1)^n 4}{4-9n^2}\right] = \frac{6\sqrt{3}(-1)^n}{\pi(4-9n^2)}
\end{aligned}$$

(As a check, note that this formula agrees with a_0 when $n = 0$.)

(c) $h(x) = \sin(2x - \pi/3)$, $[-1, 1]$.

Solution: First use the angle addition formula to write

$$\sin(2x - \frac{\pi}{3}) = \sin(2x)\cos\frac{\pi}{3} - \cos(2x)\sin\frac{\pi}{3} = \frac{1}{2}\sin(2x) - \frac{\sqrt{3}}{2}\cos(2x)$$

It remains to compute the Fourier coefficients $\{a_n^s, b_n^s\}$ of $\sin 2x$ and $\{a_n^c, b_n^c\}$ of $\cos 2x$, respectively, over the interval $[-1, 1]$. For $\sin(2x)$, an odd function, all the cosine terms vanish, so $a_n^s = 0$ for $n = 0, 1, \ldots$ Likewise, for the even function $\cos(2x)$ all the sine terms vanish, so $b_n^c = 0$ for $n = 1, 2, \ldots$ Thus the Fourier coefficients of h will be

$$a_n = \frac{1}{2}a_n^s - \frac{\sqrt{3}}{2}a_n^c = -\frac{\sqrt{3}}{2}a_n^c; \qquad b_n = \frac{1}{2}b_n^s - \frac{\sqrt{3}}{2}b_n^c = \frac{1}{2}b_n^s.$$

Finally,

$$a_n^c = \int_{-1}^{1}\cos(2x)\cos(n\pi x)\,dx = \frac{4(-1)^n\sin 2}{4-\pi n^2},$$

and

$$b_n^s = \int_{-1}^{1} \sin(2x)\sin(n\pi x)\,dx = \frac{2n\pi(-1)^n \sin 2}{4-\pi n^2},$$

using the formulas on p.166 and p.186 of the textbook.

Q.3 Fix $L > 0$. Show that

$$\frac{L}{2} - x = \frac{L}{\pi}\sum_{j=1}^{\infty} \frac{1}{j}\sin\frac{2j\pi x}{L}, \qquad 0 < x < L.$$

Solution: The function $f(x) = L/2 - x$ defined on $0 < x < L$, when extended to be L-periodic, is antisymmetric, so it has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{L}$$

with b_n given by the formulas on p.186 and p.180:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L (\frac{L}{2} - x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[\frac{L^2}{2\pi n}\right] = \frac{L}{\pi n},$$

which gives the result.

Note: The integral may be done by hand using integration by parts, or with a computer algebra program like Macsyma where the command would be

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integrate((L/2-x)*sin(2*%pi*n*x/L),x,0,L);
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Q.4 Verify that each pair of functions below is orthogonal on the given interval $[a, b]$ using the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$.

(a) $f(x) = \sin 2x$, $g(x) = \cos 3x$, $[-\pi, \pi]$.

Solution: Use the identity on p.166 of the textbook:

$$\int_{-\pi}^{\pi} \sin 2x \cos 3x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(2+3)x + \sin(2-3)x]\,dx = 0,$$

since $[-\pi, \pi]$ is a period interval, and $\sin kx$ integrates to zero on a period intervals whenever k is an integer.

(b) $f(x) = \sin 2x$, $g(x) = \sin 4x$, $[0, \pi]$.

Solution: Use the identity $\sin nx \sin jx = \frac{1}{2}[\cos(n-j)x - \cos(n+j)x]$ to compute

$$\int_0^{\pi} \sin 2x \sin 4x\,dx = \frac{1}{2}\int_0^{\pi} [\cos 2x - \cos 6x]\,dx = 0,$$

since the antiderivatives $(-\sin 2x)/2$ and $(-\sin 6x)/6$ are zero at $x = 0$ and $x = \pi$.

(c) $f(x) = x^2$, $g(x) = x^3$, $[-1, 1]$.

Solution: Direct computation:

$$\int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} x^5\,dx = \left.\frac{x^6}{6}\right|_{-1}^{1} = 0.$$

Alternatively, note that $f(x)g(x)$ is an odd function so its integral vanishes on any symmetric interval $[-a, a]$.

(d) $f(x) = x$, $g(x) = \cos x$, $[-2, 2]$.

Solution: As in part c, since $f(x)g(x) = x\cos x$ is an odd function, so its inegral vanishes on any symmetric interval $[-a, a]$, in particular for $[-2, 2]$.

Q.5 Prove the following identities for functions $f, g \in L^2$:

(a) Parallelogram law: $2\|f\|^2 + 2\|g\| = \|f + g\|^2 + \|f - g\|^2$.

Solution: Expand from the definition $\|f\|^2 = \langle f, f\rangle$, using the linearity and symmetry of $\langle\cdot,\cdot\rangle$:

$$\begin{aligned} \|f+g\|^2 &= \langle f+g, f+g\rangle = \|f\|^2 + 2\langle f, g\rangle + \|g\|^2 \\ \|f-g\|^2 &= \langle f-g, f-g\rangle = \|f\|^2 - 2\langle f, g\rangle + \|g\|^2 \\ \implies \|f+g\|^2 + \|f-g\|^2 &= 2\|f\|^2 + 2\|g\|^2 \end{aligned}$$

(b) Pythagoras' theorem and its converse: f, g are orthogonal if and only if $\|f - g\|^2 = \|f\|^2 + \|g\|^2$.

Solution: From part a, $\|f\|^2 + \|g\|^2 - \|f - g\|^2 = 2\langle f, g\rangle = 0$ if and only if $\langle f, g\rangle = 0$, if and only if f, g are orthogonal.

(c) Cauchy-Schwarz-Bunyakovsky inequality: $|\langle f, g\rangle| \le \|f\| \cdot \|g\|$.

(Hint: let $\phi(t) = \|f + tg\|^2$ and find the minimum of ϕ, noting that it must be greater than or equal to zero.)

Solution: Following the hint, expand

$$\phi(t) = \|f + tg\|^2 = \|f\|^2 + 2\langle f, g\rangle t + \|g\|^2 t^2 \ge 0$$

for all t, since norms are nonnegative. This is a quadratic function in t whose t^2 coefficient is positive, so its global minimum is the vertex $t_0 = -\langle f, g\rangle / \|g\|^2$. Substituting this value into nonnegative ϕ gives

$$0 \le \phi(t_0) = \|f\|^2 - 2\langle f, g\rangle \langle f, g\rangle / \|g\|^2 + \|g\|^2 \langle f, g\rangle^2 / \|g\|^4$$

Solving gives $\langle f, g\rangle^2 \le \|f\|^2\|g\|^2$, and taking square roots gives the result.