MATH 217 – WORKSHEET 06

Q.1 For each of the following ODEs, verify that the origin is a regular singular point and find two linearly independent Frobenius series solutions.

(a) $2xy'' + (3-x)y' - y = 0$

(b) $2x^2y'' + xy' - (x+1)y = 0$

Q.2 Find the Fourier series for the functions below:

(a) $f(x) = 0$ if $-\pi \le x < 0$, while $f(x) = \sin x$ if $0 \le x < \pi$.

(b) $f(x) = 0$ if $-\pi \le x < 0$, while $f(x) = \cos x$ if $0 \le x < \pi$.

Q.3 Find the Fourier series for the functions below:

(a) $f(x) = -1$ if $-\pi \le x < 0$, while $f(x) = +1$ if $0 \le x < \pi$.

(b) $g(x) = x - \frac{\pi}{2}$ if $-\pi \le x < 0$, while $g(x) = \frac{\pi}{2} - x$ if $0 \le x < \pi$.

Q.4 Let f be the 2π periodic function defined on $[-\pi, \pi)$ by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x^2, & 0 \le x < \pi \end{cases}$$

(a) Find the Fourier series for f.

(b) Use Dirichlet's theorem (Th.7.2.7, p.177 in the textbook) with the results from part (a) at $x = 0$ to prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

(c) Use Dirichlet's theorem with the part (a) series at $x = \pi$ to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

(d) Derive part (c) from part (b). (Hint: add $2\sum_n (1/[2n]^2)$ to both sides.)

Q.5 Determine whether the following functions are odd, even, or neither.

$$x^5 \sin x, \quad x^2 \sin 2x, \quad e^x, \quad (\sin x)^3, \quad \sin x^2, \quad \cos(x+x^3), \quad \frac{\sin x}{x}, \quad x+x^2+x^3, \quad \ln \frac{1+x}{1-x}$$

Q.6 Let $f(x) = \pi/4$ be the constant function.

(a) Show that the sine series for f is

$$\frac{\pi}{4} = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots, \qquad 0 < x < \pi.$$

(b) Let $x = \pi/2$ in part (a) and deduce an infinite sum formula.

(c) Find the cosine series for f.