MATH 217 – WORKSHEET 05

Q.1 If $p \neq 0$ and p is not a positive integer, show that the power series

$$\sum_{j=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-j+1)}{j!}\, x^j$$

converges for $|x| < 1$ and diverges for $|x| > 1$.

Q.2 By the geometric sum formula, we have the power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad \text{and} \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

and these converge absolutely if $|x| < 1$. Show the following:

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

(b) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(c) $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Q.3 For each of the following ODEs, find a power series solution of the form $\sum_j a_j x^j$. Then solve the equation by some method from Chapter 2.

(a) $y' = xy$

(b) $y' - y = x$

(c) $xy' = y$

(d) $y' - (1/x)y = x^2$

Q.4 In each of the following problems, verify that 0 is an ordinary point. Then find the power series solution of the ODE.

(a) $y'' - xy' - y = 0$

(b) $(1 + x^2)y'' + xy' + y = 0$

Q.5 *Chebyshev's equation* is $(1-x^2)y'' - xy' + p^2 y = 0$, where p is constant.

(a) Find two linearly independent solutions valid for $|x| < 1$.

(b) Show that if $p = n$ is a nonnegative integer, then there is a polynomial solution of degree n. (These solutions, when multiplied by suitable normalizing constants, are called *Chebyshev polynomials*.)