Math 426

Solutions to Practice Midterm 2

1. Let f be holomorphic on the disc $D(P,r)$ with bound $|f(z)| \le M$. Then, for any nonnegative integer m,

$$\left| \frac{d^m}{dz^m} f(P) \right| \le \frac{M \cdot m!}{r^m}.$$

2. Let f be entire and bounded by M. If $P \in \mathbb{C}$ then, by the Cauchy estimates on $D(P,r)$,

$$|f'(P)| \le \frac{M \cdot 1!}{r^1} = \frac{M}{r}.$$

Letting $r \to \infty$ gives $f'(P) = 0 \ \forall P$. Hence f is constant.

3. $f(z) = \sin z \cos z = \frac{1}{2} \sin 2z = \left(2z - \frac{(2z)^3}{3!} + - \cdots\right) \cdot \frac{1}{2}$

The coefficient of z^3 is $-\frac{4}{6}$.

4. $\text{residue} = z \cdot \left. \frac{(z-1)z}{\sin^2 z} \right|_{z=0} = -1$

5. $G(z) = e^{1/z}$ has essential singularity at the origin. Hence f has an essential singularity at ∞.

6. Let f be holomorphic on a region $U \subseteq \mathbb{C}$. If $\exists\ p \in U$ so that

$$|f(p)| \geq |f(z)| \text{ for all } z \in U$$

then f is constant.

7. Let f be a conformal self-map of the disc. Then f is the composition of a

rotation $z \mapsto e^{i\theta_0} z$

and a

Möbius transformation $z \to \frac{z-a}{1-\bar{a}z}$?

some $a \in \mathbb{C}$, $|a| < 1$.

8. Let $g(z) = 5z$. Then, on ∂D,

$$|f(z) - g(z)| = |z^4 + 1| \leq |z|^4 + 1 < |g(z)| \text{ on } \partial D.$$

So f has the same number of zeros as g in D (by Rouché) and that number is 1.

9. Let f be holomorphic on D with

$$|f(z)| \leq |z| \quad \forall z \in D$$

and $f(a) = b$ for some $a, b \in D$.

Then

$$|f'(a)| \leq \frac{1-|b|^2}{1-|a|^2}.$$

Also, if $f(a_1) = b_1$, $f(a_2) = b_2$, then

$$\left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| \leq \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|.$$

Furthermore, if

$$|f'(a)| = \frac{1-|b|^2}{1-|a|^2} \quad \text{or} \quad \left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| = \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|$$

with $a_1 \neq a_2$ then f is a conformal self-map of the disc D.

10. $$f(z) = \frac{\frac{1}{2}\sin 2z}{z^2} = \frac{\frac{1}{2}\left(2z - \frac{8z^3}{3!} + \frac{32z^5}{5!} - \frac{128z^7}{7!} + - \cdots\right)}{z^2}$$

$$= z - \frac{8}{6}z + \frac{32}{120}z^3 - \frac{128}{5040}z^5 + - \cdots.$$