SOLUTIONS

Math 416
Krantz

Spring, 2022
March 3, 2022

FIRST MIDTERM

General Instructions: Read the statement of each problem carefully. On each problem you should show your work. If you only write the answer then you will not receive full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points.

(10 points) **1.** Calculate the multiplicative inverse of $z = 3 + 2i$.

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{3-2i}{9+4} = \frac{3}{13} - \frac{2}{13}i$$

(10 points) **2.** Calculate this derivative:

$$\frac{\partial}{\partial \overline{z}}\left(z \cdot \overline{z}^2 + \overline{z}\cos z\right).$$

$= 2z\overline{z} + \cos z$

(10 points) **3.** Show that the function $u(x, y) = x^2 - y^2$ is harmonic. Find the real-valued harmonic conjugate function v so that $u + iv$ is holomorphic.

$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = 2 - 2 = 0.$

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y$

$v(x,y) = 2xy + \varphi(y)$ $\quad 2x = \frac{\partial v}{\partial y} = 2x + \varphi'(y) \Rightarrow \varphi'(y) = 0$

$\varphi(y) = C.$

Hence $v(x,y) = 2xy + C$

Holo. fcn. is $(x^2 - y^2) + i(2xy) + iC$

$= z^2 + iC$

(10 points) **4.** Explain why every holomorphic function is harmonic but the converse is not true.

If f is holomorphic then $\frac{\partial f}{\partial \bar{z}} \equiv 0$ so $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f \equiv 0$. Hence $\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0$.

Write $f = u + iv$. Then $\Delta(u+iv) = 0$ so $\Delta u + i\Delta v = 0$.

Thus $\Delta u \equiv 0$, $\Delta v \equiv 0$. So u, v harmonic.

Converse not true because $2x$ is harmonic but $2x = z + \bar{z}$ is <u>not</u> holomorphic.

(10 points) **5.** What does the Cauchy integral formula say? To what class of functions does it apply? To what class of curves does it apply?

Let f be holomorphic on a region $U \subseteq \mathbb{C}$.

Let $\overline{D}(p, r) \subseteq U$. Then, for $z \in D(p, r)$,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(p,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

More generally, the result is true for any single closed curve γ in U that can be continuously deformed to a point and z interior to γ.

(10 points) **6.** What does the Cauchy integral theorem say? To what class of functions does it apply? To what class of curves does it apply?

If $U \subseteq \mathbb{C}$ is a region and γ is a closed curve in U that can be continuously deformed to a point then

$$\oint_\gamma f(z)\,dz = 0.$$

(10 points) **7.** State Morera's theorem.

If $U \subseteq \mathbb{C}$ is a region, f is C^1 on U, and $\oint_\gamma f(z)\,dz = 0$ for every closed curve γ in U, then f is holomorphic on U.

(10 points) **8.** Let $f(z) = z^2 + z$ and $\gamma(t) = \cos t + i \sin t$, $0 \le t \le 2\pi$. Calculate

$$\oint_\gamma f(z)\, dz\,.$$

$$\oint_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t))\cdot\gamma'(t)\,dt = \int_0^{2\pi} [(\cos t + i\sin t)^2 + (\cos t + i\sin t)] \cdot(-\sin t + i\cos t)\,dt$$

$$= \int_0^{2\pi} [\cos^2 t - \sin^2 t + 2i\cos t\sin t + \cos t + i\sin t]\cdot(-\sin t + i\cos t)\,dt$$

$$= \int_0^{2\pi} -\cos^2 t\sin t + i\cos^3 t + \sin^3 t - i\cos t\sin^2 t - 2i\cos t\sin^2 t$$
$$- 2\cos^2 t\sin t - \sin t\cos t + i\cos^2 t - i\sin^2 t - \sin t\cos t\,dt$$

$$= \int_0^{2\pi} -3\cos^2 t\sin t - 3i\cos t\sin^2 t - 2\sin t\cos t + i\cos 2t + i\cos^3 t$$
$$+ \sin^3 t\,dt = \cdots = 0.$$

(10 points) **9.** The Cauchy integral theorem fails for the holomorphic function $f(z) = 1/z$ on the annulus $A = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. Explain why.

Let $\gamma(t) = \cos t + i\sin t = e^{it}$, $0 \le t \le 2\pi$.

Then $\oint_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\cdot ie^{it}\,dt = \int_0^{2\pi} i\,dt$

$= 2\pi i$.

This happens because f has a singularity at 0. And of course γ <u>cannot</u> be continuously deformed to a point in A.

(10 points) **10.** Let U be an open region in the complex plane. Explain what it means for a function to be C^k on U.

We say that f is C^k on U if all partial derivatives of f of orders up to and including k exist and are continuous on $\overline{U}$.