Solutions to HW6

pp. 129 - 133

1. $f(z) = \cot z$

$$\operatorname{Res}_f(0) = \frac{1}{0!}\left(\frac{\partial}{\partial z}\right)^0 \left(\frac{(z-p)\cos z}{\sin z}\right)\Big|_{z=0}$$

$$\overset{\text{l'Hôpital}}{=} \frac{1\cdot\cos z + (z-p)(-\sin z)}{\cos z}\Big|_{z=0}$$

$$= \frac{1+0}{1} = 1$$

3. $f(z) = e^{1/z} = \sum_{j=0}^{\infty} \frac{(1/z)^j}{j!} = \sum_{j=-\infty}^{0} \frac{(1/z)^{-j}}{|j|!}$

$$= \sum_{j=-\infty}^{0} \frac{z^j}{|j|!}.$$

The term with index -1 is $\frac{z^{-1}}{1!}$. The residue is thus 1.

5. $g(z) = \sin\frac{1}{z} = \sum_{j=0}^{\infty} \frac{(-1)^j (1/z)^{2j+1}}{(2j+1)!}$

$$= \sum_{j=-\infty}^{0} \frac{(-1)^{-j}(1/z)^{1-2j}}{|-2j+1|!}$$

$$= \sum_{j=-\infty}^{0} \frac{(-1)^j z^{2j-1}}{|-2j+1|!}$$

The term with index -1 is $\frac{(-1)^0 z^{-1}}{1!}$.

The residue is thus 1.

11 a) $\frac{1}{2\pi i}\oint_{\partial D(0,5)} \frac{z}{(z+1)(z+2i)}\,dz \qquad f(z)=\frac{z}{(z+1)(z+2i)}$

$= \operatorname{Res}_f(-1) + \operatorname{Res}_f(-2i) \qquad (*)$

$\operatorname{Res}_f(-1) = (z+1)\cdot\frac{z}{(z+1)(z+2i)}\Big|_{z=-1}$

$= \frac{z}{z+2i}\Big|_{z=-1} = \frac{-1}{-1+2i} = \frac{-1(-1-2i)}{(-1+2i)(-1-2i)}$

$= \frac{1+2i}{5} = \frac{1}{5}+\frac{2}{5}i$

$\operatorname{Res}_f(-2i) = (z+2i)\cdot\frac{z}{(z+1)(z+2i)}\Big|_{z=-2i}$

$= \frac{z}{z+1}\Big|_{z=-2i}$

$= \frac{-2i}{-2i+1} = \frac{-2i(1+2i)}{(1-2i)(1+2i)}$

$= \frac{4-2i}{5} = \frac{4}{5}-\frac{2}{5}i.$

So $(*) = \left(\frac{1}{5}+\frac{2}{5}i\right)+\left(\frac{4}{5}-\frac{2}{5}i\right) = 1.$

f) $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz \qquad f(z) = \frac{\cos z}{z^2(z+1)^2(z+i)}$

$= \operatorname{Res}_f(0)\cdot \operatorname{Ind}_\gamma(0) + \operatorname{Res}_f(-1)\operatorname{Ind}_\gamma(-1)$

$+ \operatorname{Res}_f(-i)\operatorname{Ind}_\gamma(-i)$

$$= \operatorname{Res}_f(0)\cdot 1 + \operatorname{Res}_f(-1)\cdot 1 + \operatorname{Res}_f(-i)\cdot 1$$

Now

$$\operatorname{Res}_f(0) = \frac{1}{(2-1)!}\left(\frac{\partial}{\partial z}\right)^{2-1}\left((z-0)^2\cdot\frac{\cos z}{z^2(z+1)^2(z+i)}\right)\Big|_{z=0}$$

$$= \frac{\partial}{\partial z}\left(\frac{\cos z}{(z+1)^2(z+i)}\right)\Big|_{z=0}$$

$$= \frac{(z+1)^2(z+i)(-\sin z) - \cos z\left[2(z+1)(z+i) + (z+1)^2\cdot 1\right]}{(z+1)^4(z+i)^2}\Big|_{z=0}$$

$$= \frac{1^2\cdot i\cdot 0 - 1\cdot(2(1)(i)+1)}{1\cdot i^2}$$

$$= \frac{-2i-1}{-1} = 2i+1$$

$$\operatorname{Res}_f(-1) = \frac{1}{(2-1)!}\left(\frac{\partial}{\partial z}\right)^{2-1}\left((z+1)^2\cdot\frac{\cos z}{z^2(z+1)^2(z+i)}\right)\Big|_{z=-1}$$

$$= \frac{\partial}{\partial z}\left(\frac{\cos z}{z^2(z+i)}\right)\Big|_{z=-1}$$

$$= \frac{z^2(z+i)\cdot(-\sin z) - \cos z\left(2z(z+i) + z^2\cdot 1\right)}{z^4(z+i)^2}\Big|_{z=-1}$$

$$= \frac{1\cdot(-1+i)(-\sin(-1)) - \cos(-1)(-2(-1+i)+1)}{1\cdot(-1+i)^2}$$

$$= \frac{(\sin 1)\cdot(-1+i) - (\cos 1)\cdot(3-2i)}{(-1+i)^2}$$

$$= \frac{(\sin 1)\cdot(-1+i) - (\cos 1)\cdot(3-2i)}{-2i}$$

$$\operatorname{Res}_f(-i) = \frac{1}{(2-1)!}\left(\frac{\partial}{\partial z}\right)^{2-1}\left((z+i)\cdot\frac{\cos z}{z^2(z+1)^2(z+i)}\right)\Big|_{z=-i}$$

$$= \frac{\partial}{\partial z}\left(\frac{\cos z}{z^2(z+1)^2}\right)\Big|_{z=-i}$$

$$= \frac{z^2(z+1)^2(-\sin z) - \cos z\left(2z(z+1)^2 + z^2 2(z+1)\right)}{z^4(z+1)^4}\Big|_{z=-i}$$

$$= \frac{-1(1-i)^2(-\sin(-i)) - \cos(-i)\left(-2i(1-i)^2 - 1\cdot 2(1-i)\right)}{1\cdot(1-i)^4}$$

$$= \frac{-(1-i)^2\cdot\sin i - \cos(i)\left(\overline{+2i} - (-2i) - 2 + 2i\right)}{(1-i)^4}$$

$$= \frac{+2i\cdot\sin i - \cos i\cdot(-6+2i)}{-4}$$

$$\int_\gamma \frac{1}{2\pi i}\oint f(z)\,dz = 2i + 1 + \frac{(\sin 1)\cdot(-1+i) - (\cos 1)\cdot(3-2i)}{-2i} + \frac{2i\sin i + \cos i\,(6-2i)}{-4}$$

i. $\dfrac{1}{2\pi i}\oint_\gamma f(z)\,dz$ $\qquad f(z) = \tan z$

$$= \operatorname{Res}_f\left(\tfrac{\pi}{2}\right)\cdot \operatorname{Ind}_\gamma\left(\tfrac{\pi}{2}\right) + \operatorname{Res}_f\left(-\tfrac{\pi}{2}\right)\cdot \operatorname{Ind}_\gamma\left(-\tfrac{\pi}{2}\right)$$

$$= \operatorname{Res}_f\left(\tfrac{\pi}{2}\right)\cdot 1 + \operatorname{Res}_f\left(-\tfrac{\pi}{2}\right)\cdot(-1)$$

Now

$$\operatorname{Res}_f\left(\frac{\pi}{2}\right) = \frac{1}{0!}\left(\frac{\partial}{\partial z}\right)^0\left(\left(z-\frac{\pi}{2}\right)\cdot\frac{\sin z}{\cos z}\right)\Big|_{z=\frac{\pi}{2}}$$

$$= \frac{1\cdot\sin z + \left(z-\frac{\pi}{2}\right)\cdot\cos z}{-\sin z}\Big|_{z=\frac{\pi}{2}}$$

$$= \frac{1+0\cdot 0}{-1} = -1$$

$$\operatorname{Res}_f\left(-\frac{\pi}{2}\right) = \frac{1}{0!}\left(\frac{\partial}{\partial z}\right)^0\left(\left(z+\frac{\pi}{2}\right)\frac{\sin z}{\cos z}\right)\Big|_{z=-\frac{\pi}{2}}$$

$$= \frac{1\cdot\sin z + \left(z+\frac{\pi}{2}\right)\cdot\cos z}{-\sin z}\Big|_{z=-\frac{\pi}{2}}$$

$$= \frac{-1+0-0}{1} = -1$$

So $\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = (-1)\cdot 1 + (-1)\cdot(-1) = 0.$

13. If f has a removable singularity at P, then f continues to be holomorphic on $D(P,r)$. Then $g\circ f$ is holomorphic on $U\cup\{f(P)\}$ just by applying $\frac{\partial}{\partial z}$ to $g\circ f$.

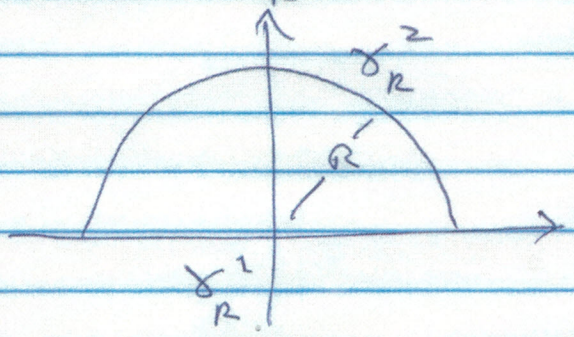
pp. 148 - 150

1. $\int_0^\infty \frac{1}{1+x^2}\,dx$

Instead we calculate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$. We in fact integrate

$$f(z) = \frac{1}{1+z^2}$$

over the contour γ_R:



The relevant pole of f is at i.

$$\frac{1}{2\pi i}\oint_{\gamma_R} f(z)\,dz = \operatorname{Res}_f(i) = (z-i)\cdot\frac{1}{1+z^2}\Big|_{z=i}$$

$$= \frac{1}{z+i}\Big|_{z=i} = \frac{1}{2i} = \frac{-i}{2}.$$

Also $\oint_{\gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx \to \int_{-\infty}^{\infty} f(x)\,dx$

as $R \to +\infty$.

And

$$\left| \oint_{\gamma_R^2} f(z)\,dz \right| \leq \pi R \max_{\gamma_R^2} |f(z)|$$

$$\leq \frac{\pi R}{R^2 - 1} \to 0 \quad \text{si } R \to +\infty,$$

In summary

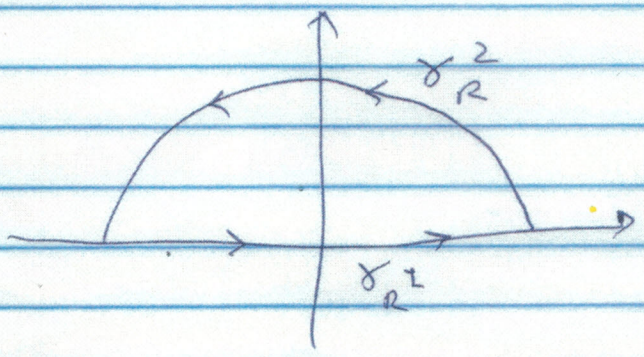
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx + 0 = \lim_{R \to \infty} \oint_{\gamma_R} f(z)\,dz = 2\pi i \left(-\frac{i}{2}\right) = \pi,$$

$$\Rightarrow \int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

$$2 \cdot \int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}\,dx$$

We will integrate $f(z) = \dfrac{e^{iz}}{1+z^4}$

over the curve γ_R:



The relevant poles of f are at

$\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

We see that

$$\oint_{\gamma_R^1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^4}\,dx \to \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4}\,dx \quad \text{as } R \to +\infty$$

and

$$\left|\oint_{\gamma_R^2} f(z)\,dz\right| \le \pi R \cdot \max_{\gamma_R^2} |f(z)|$$

$$\le \frac{\pi R}{R^4 - 1} \to 0 \quad \text{as } R \to +\infty,$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^4}\,dx = \lim_{R\to\infty} \oint_{\gamma} f(z)\,dz$$

$$= 2\pi i \left(\operatorname{Res}_f\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + \operatorname{Res}_f\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right)$$

Now

$$\operatorname{Res}_f\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \left(z - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right) \cdot \frac{e^{iz}}{1+z^4}\Bigg|_{z=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}}$$

$$= \frac{1 \cdot e^{iz} + \left(z - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right)e^{iz}}{4z^3}\Bigg|_{z=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}}$$

$$= \frac{e^{\frac{i}{\sqrt{2}}} e^{-\frac{1}{\sqrt{2}}}}{4\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^3}$$

$$\text{Res}_f\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \left(z - \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right) \cdot \frac{e^{iz}}{1+z^4}\Bigg|_{z=-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}}$$

$$= \frac{1 \cdot e^{iz} + \left(z - \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right)e^{iz}}{4z^3}\Bigg|_{z=-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}}$$

$$= \frac{e^{-i/\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}}{4\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^3}$$

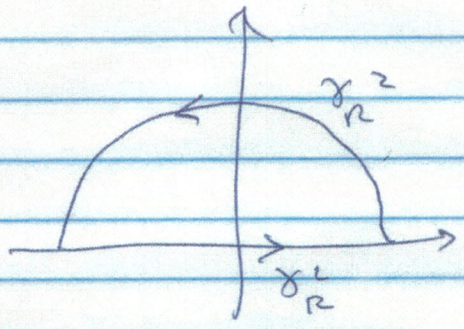
$$\int_{-\infty}^{\infty} \frac{\cos x}{2+x^4}\,dx = \text{Re}\left[2\pi i\left(\frac{e^{i/\sqrt{2}} e^{-1/\sqrt{2}}}{4\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^3} + \frac{e^{-i/\sqrt{2}} e^{-1/\sqrt{2}}}{4\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)^3}\right)\right].$$

I will not bother to simplify.

11. $\int_{-\infty}^{\infty} \frac{x^4}{1+x^{10}}\,dx$

let $f(z) = \frac{z^4}{1+z^{10}}$.

Use the contour γ_R:



For $R > 1$, relevant poles are at

$e^{i\pi/10}$, $e^{i3\pi/10}$, $e^{i5\pi/10}$, $e^{i7\pi/10}$, $e^{i9\pi/10}$.

Now

$$\oint_{\gamma_R^1} f(z)\,dz = \int_{-R}^{R} \frac{x^4}{1+x^{10}}\,dx \to \int_{-\infty}^{\infty} \frac{x^4}{1+x^{10}}\,dx$$

as $R \to +\infty$.

$$\left| \oint_{\gamma_R^2} f(z)\,dz \right| \le \pi R \cdot \max_{\gamma_R^2} |f(z)| \le \frac{\pi R}{R^{10}-1} \to 0$$

as $R \to +\infty$.

It remains to calculate the residues.

$$\operatorname{Res}_f\left(e^{i\frac{\pi}{10}}\right) = \left(z - e^{i\frac{\pi}{10}}\right) \cdot \frac{z^4}{1+z^{10}}\Bigg|_{z=e^{i\frac{\pi}{10}}}$$

$$= \frac{1 \cdot z^4 + \left(z - e^{i\frac{\pi}{10}}\right) \cdot 4z^3}{10z^9}\Bigg|_{z=e^{i\frac{\pi}{10}}}$$

$$= \frac{z^4}{10z^9}\Bigg|_{z=e^{i\frac{\pi}{10}}}$$

$$= \frac{1}{10} z^{-5}\Big|_{z=e^{i\frac{\pi}{10}}}$$

$$= \frac{1}{10} e^{-i\pi/2} = \frac{-i}{10}$$

$$\operatorname{Res}_f\left(e^{i\frac{3\pi}{10}}\right) = \left(z - e^{i\frac{3\pi}{10}}\right) \cdot \frac{z^4}{1+z^{10}}\Bigg|_{z=e^{i\frac{3\pi}{10}}}$$

$$= \frac{1 \cdot z^4 + \left(z - e^{i\frac{3\pi}{10}}\right) \cdot 4z^3}{10z^9}\Bigg|_{z=e^{i\frac{3\pi}{10}}}$$

$$= \frac{z^4}{10z^9}\Bigg|_{z=e^{i\frac{3\pi}{10}}}$$

$$= \frac{1}{10} z^{-5}\Big|_{z=e^{i\frac{3\pi}{10}}}$$

$$= \frac{1}{10} e^{-\frac{3\pi i}{2}} = \frac{i}{10}$$

$$\operatorname{Res}_f\left(e^{i\frac{5\pi}{10}}\right) = \left(z - e^{i\frac{5\pi}{10}}\right)\cdot\frac{z^4}{1+z^{10}}\bigg|_{z=e^{i\frac{5\pi}{10}}}$$

$$= \frac{1\cdot z^4 + \left(z - e^{i\frac{5\pi}{10}}\right)\cdot 4z^3}{10z^9}\bigg|_{z=e^{i\frac{5\pi}{10}}}$$

$$= \frac{z^4}{10z^9}\bigg|_{z=e^{i\frac{5\pi}{10}}}$$

$$= \frac{1}{10}z^{-5}\bigg|_{z=e^{i\frac{5\pi}{10}}}$$

$$= \frac{1}{10}e^{-i\frac{5\pi}{2}} = -i/10$$

Similarly,

$$\operatorname{Res}_f\left(e^{i\frac{7\pi}{10}}\right) = i/10$$

$$\operatorname{Res}_f\left(e^{i\frac{9\pi}{10}}\right) = -i/10$$

$$\int_{-\infty}^{\infty}\frac{x^4}{1+x^{10}}dx = \lim_{R\to\infty}\oint_{\gamma_R} f(z)dz$$

$$= 2\pi i\left(-\frac{i}{10} + \frac{i}{10} - \frac{i}{10} + \frac{i}{10} - \frac{i}{10}\right)$$

$$= 2\pi i\left(-\frac{i}{10}\right) = \frac{\pi}{5}.$$

25. Let $f(\zeta) = \dfrac{\cot(\pi\zeta)}{\zeta^3+2}$

Then f has poles at $z = j$, $j \in \mathbb{Z}$.

$$\operatorname{Res}_f(j) = \left.\frac{(\zeta - j)\cdot \frac{\cos(\pi\zeta)}{\sin \pi\zeta}}{\zeta^3+2}\right|_{\zeta = j}$$

$$= \left.\frac{\cos(\pi\zeta)/(-1)^j}{\zeta^3+2}\right|_{\zeta=j}$$

$$= \frac{(-1)^j/(-1)^j}{j^3+2} = \frac{1}{j^3+2}.$$

Thus, with $T_n \to$ is Figure 8.12 in the text,

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{T_n} f(\zeta)\,d\zeta = \sum_{j=-\infty}^{\infty} \frac{1}{j^3+2}$$

$\downarrow$

0 as $n \to \infty$. So

$$\sum_{j=-\infty}^{\infty} \frac{1}{j^3+2} = 0.$$

pp. 157 - 159

1. If p is a polynomial of degree > 1 and zero free, then

$$\frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{p'(z)}{p(z)}\, dz = 0$$

$\forall\, r > 0$. As $r \to +\infty$ the expression on the left tends to

$$\frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{k a_k z^{k-1}}{a_k z^k}\, dz,$$

where

$$p(z) = a_0 + a_1 z + \cdots + a_k z^k.$$

This last

$$= \frac{1}{2\pi i} \oint_{\partial D(0,r)} \frac{k}{z}\, dz = k \neq 0.$$

Contradiction.

3. Let $f(z) = z^2 + 1$ on $D(0,1)$.

Then f has no zeros, but

$$f'(z) = 2z$$

has 1 zero.

6. If $f'(p) = 0$ Then

$$f(z) = f(p) + \frac{f''(p)}{2!}(z-p)^2 + \cdots$$

We may as well suppose That $f(p) = 0$.

Then f is a square (or worse) function which is

2-to-1.